RESEARCH ON TEACHING MATHEMATICS: MAKING SUBJECT MATTER KNOWLEDGE PART OF THE EQUATION*

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Introduction

Subject matter understanding and its role in teaching mathematics are the focus of this paper. Although few would disagree with the assertion that, in order to teach mathematics effectively, teachers must understand mathematics themselves, past efforts to show the relationship of teachers' mathematical knowledge to their teaching of mathematics have been largely unsuccessful. How can this be? My purpose here is to unravel this intuitively indisputable yet empirically unvalidated requirement of teaching by revisiting what it means to "understand mathematics" and the role played by such understanding in teaching.

The thesis of this paper is that teachers' subject matter knowledge interacts with their assumptions and explicit beliefs about teaching and learning, about students, and about context to shape the ways in which they teach mathematics to students. There are three parts to the development of this argument. First, I briefly analyze past investigations of the role of teachers' subject matter knowledge in teaching mathematics. Next, I unpack the concept of subject matter knowledge for teaching mathematics and illustrate what is entailed in finding out what teachers know. The last section presents three cases of teaching multiplication and analyzes how each teacher's understanding of mathematics figures in her teaching.

To provide a context, I begin by tracing briefly the history of efforts to identify and understand the critical variables in effective mathematics teaching. This history is inevitably nested within the larger story of research on teaching, for it is only recently that many researchers have begun to think about teaching as subject-matter specific.

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Research on Teaching Mathematics: Coming Full Circle on Subject Matter Knowledge

Through three phases of research on teaching, teachers' subject matter knowledge has figured, faded, and reappeared as a key influence on the teaching of mathematics. Driven by common sense and conventional wisdom about teaching, the earliest research compiled characteristics of teachers whom others perceived as effective (Medley, 1979). The second phase of research attempted to establish connections between what teachers do and what their students learn. In the most recent phase, researchers have investigated teacher thinking.

What Are Effective Teachers Like?

Researchers began by collating the characteristics of good teachers. Based on pupils' assessments of their best teachers, these studies reported that good teachers were enthusiastic, helpful, and strict. Students also said that the best teachers knew the subject matter better (e.g., Hart, 1934). Although such findings seemed intuitively valid, the early studies did not empirically test the influence of "good" teachers' characteristics on what they did or what their students actually learned.

Recognizing the weakness of such claims, researchers began defining "effective teaching" as teaching that results in measurable student learning. In the most ambitious effort to identify teacher characteristics associated with student achievement in mathematics, the National Longitudinal Study of Mathematical Abilities followed 112,000 students from over 1500 schools in 40 states during the 1960s. Twenty teacher characteristics were studied, including years of teaching experience, credits in mathematics, having a major or minor in mathematics, personal enjoyment of mathematics, and philosophical orientation to learning. Overall, neither teacher background characteristics nor teacher attitudes were strongly related to student learning; significant positive relationships were found in fewer than 30 percent of the possible cases. No single teacher characteristic proved to be "consistently and significantly correlated with student achievement" (Begle and Geeslin, 1972). Begle (1979) concluded from these results that many widely held beliefs about good teaching "are false, or at the very best rest on shaky foundations" (p. 54).

One of these beliefs was the notion that the more one knows about one's subject, the more effective one can be as a teacher. "The empirical literature suggests that this belief needs drastic modification," wrote Begle (1979, p. 51). The analyses showed that students whose teachers had majored or minored in mathematics scored significantly higher in only 20 percent of the cases. The number of teacher credits in college mathematics was actually *negatively* associated with student achievement in 15 percent of the cases. Convinced by these results that "the effects of a teacher's subject matter knowledge and attitudes on student learning seem to be far less powerful than many of us assumed," Begle argued that researchers should focus their inquiries elsewhere (p. 53).

Begle's (1979) conclusion was counterintuitive. Teaching is fundamentally tied up with knowledge and the growth of knowledge (Buchmann, 1984). What sense does it make to say that what teachers know about mathematics is not a significant influence on what their students learn? Yet, in spite of the weight of common sense, the empirical results were discouraging. Few questioned the assumptions underlying the research or offered alternative interpretations. For example, is the number of courses in college-level mathematics a reasonable proxy for teachers' mathematical knowledge? What is acquired through majoring in mathematics in terms of disciplinary understandings or ideas about pedagogy? Some of what is gained through sitting in upper-level mathematics courses may in fact serve as counterproductive preparation for teaching (Kline, 1977).

What Do Effective Teachers Do?

Instead of critically appraising the reported findings, however, researchers began a new search. Driven to understand what distinguished more effective from less effective teachers, the field turned from the investigation of teacher characteristics to study generic teacher behaviors such as pacing, questioning, explanation, and praise, as well as qualities such as clarity, directness, and enthusiasm. Medley (1979) explained the basis for this shift, arguing that "it is what the teacher *does* rather than what a teacher *is* that matters" (p. 13). Most of the new studies chose to focus on elementary school teaching of mathematics and reading, because achievement in these subjects in the early grades was considered central and outcomes thought to be unambiguous to measure. Subject matter was part of the context, not the focus of the research.

Rosenshine (1979) summarizes the picture of effective instruction that emerged from this work:

> Large groups, decision making by the teacher, limited choice of materials and activities by students, orderliness, factual questions, limited exploration of ideas, drill,and high percentages of correct answers. (p. 47)

He argued that, although this picture appeared grim, such orderly, businesslike classrooms need not be cold nor humorless. Furthermore, these findings, he suggested, were primarily applicable to instruction in basic skills--reading, writing, and mathematics--and that looser approaches ("messing around") might be perfectly appropriate in other subjects. While some feared that students would enjoy school less in such tightly supervised, teacher-controlled settings, studies indicated that there was little difference on such "affective outcomes" (Peterson, 1979). Some researchers even concluded that students were more anxious in informal classrooms (Bennett, 1976; Wright, 1975).

Critical to understanding this phase of research on effective teaching are its assumptions about mathematics and the goals of teaching and learning mathematics. Taking the prevalent school curriculum as given, it assumed that elementary school mathematics consists of a body of skills to be mastered through drill and practice. Careful to disclaim the assumption that learning meant accumulating facts and principles, researchers nevertheless talked about students' mathematics learning in terms of "gains." It was not surprising that within this set of assumptions researchers found that students "learned" the most from direct explanations, seatwork, and frequent quizzes in time-efficient, quiet settings.

As they spent more time in classrooms and analyzed complicated data, many researchers became increasingly appreciative of the complexity of classrooms and of the job of teaching. They saw that teachers work with a broad range of students who come with different understandings and attitudes and who do not learn in the same ways. Teachers are also responsible for a variety of educational outcomes that require different approaches. In light of these features of the job, it was simplistic to seek a single most effective teaching approach (Clark and Yinger, 1979; Peterson, 1979). Some scholars sought to uncover optimal patterns of instruction for students with particular characteristics, or "aptitude treatment interactions" (ATI) (e.g., Brophy, 1980; Evertson, Anderson, and Brophy, 1978; Solomon and Kendall, 1976). In 1982, Tobias wrote that even ATI studies were failing to specify one mode of instruction appropriate for students with a particular set of characteristics.

How Do Teachers Understand Their Work and Decide What to Do?

Several years earlier, Gage (1977) had cautioned that "no one can ever prescribe successfully all the twists and turns to be taken as the classroom teacher uses judgment, sudden insight, sensitivity, and agility to promote learning" (p. 15). In a third significant shift in research on teaching, researchers increasingly turned away from their focus on teacher behaviors and began examining teachers' thoughts and decisions. Writing in 1979, Clark and Yinger observed that this

> new approach to the study of teaching assumes that what teachers do is affected by what they think. This approach, which emphasizes the processing of cognitive information, is concerned with the teachers' judgment, decision making, and planning. The study of the thinking processes of teachers--how they gather, organize, and interpret, and evaluate information--is expected to lead to understandings of the uniquely human processes that guide and determine their behavior. (p. 231)

In search of what makes some teachers more effective than others, researchers were hot on a new trail by redefining teaching as an activity of both thought *and* action. How do teachers decide on content and goals, select materials and approaches, in order to help different students learn a variety of concepts and skills?

It was in studying teacher thinking and decision making that teachers' knowledge and beliefs about subject matter began to reappear as potentially significant variables. For example, Shroyer (1981) studied how junior high mathematics teachers coped with student difficulties or unusual responses and found that the teachers with weaker mathematics backgrounds had more difficulty generating alternative responses to these critical moments. And, in a study of fourth-grade teachers' curricular decisions, Kuhs (1980) concluded that their conceptions of mathematics and recognition of topics influenced both what the teachers taught and how they modified curriculum materials.

Thompson (1984) investigated the influence of teachers' conceptions of mathematics on their teaching. Her findings further substantiated the notion that what teachers' know about math affects what they do. One of the teachers in her study, Lynn, described mathematics as "cut and dried": a process of following procedures and producing right answers. Lynn did not provide opportunities for her students to explore or engage in creative work; instead she emphasized memorizing and using specified procedures. In contrast, Kay, who saw mathematics as a "subject of ideas and mental processes," not a "subject of facts," emphasized problem solving and encouraged her students to make and pursue their own mathematical conjectures (Thompson, 1984, pp. 112-113).

Alerted by these and other similar findings, some researchers have returned to press on subject matter as a critical variable in teaching mathematics. However, "subject matter knowledge" in current studies is a concept of varied definition, a fact that threatens to muddy our progress in learning about the role of teachers' mathematical understanding in their teaching. The next section takes up the question of what researchers should mean by "knowledge of mathematics" in the new research on math teaching.

Breaking the Circle: Moving Away From Past Errors

Philosophical arguments (e.g., Buchmann, 1984), as well as common sense, have already persuaded us that teachers' knowledge of mathematics influences their teaching of mathematics. In the most extreme case, teachers cannot help children learn things they themselves do not understand. More subtle, and much less well understood, are the ways in which teachers' understandings shape their students' opportunities to learn. The dead end of earlier attempts to investigate the relationship of teachers' understandings to teachers' effectiveness was a consequence of the ways in which both "subject matter knowledge" and "effectiveness" were defined. With different definitions and approaches, the new research on teacher knowledge has already begun to corroborate our tenacious conviction that teachers' subject matter understanding does, after all, play a significant role in the teaching of mathematics. However, if we are to move beyond what we already believe, if this research is to help us to understand the subtler effects and to improve mathematics teaching and learning, then significant conceptual issues--about what we mean by "subject matter knowledge" or by its "role" in teaching mathematics--must be addressed.

Subject Matter Knowledge in Mathematics

Although most researchers have moved away from the earlier use of course lists or credits earned as a proxy for teachers' knowledge, how they conceptualize and study "subject matter" varies. Some researchers examine teachers' *conceptions of* or *beliefs about mathematics* (e.g., Blaire, 1981; Ernest, 1988; Ferrini-Mundy, 1986; Kuhs, 1980; Lerman, 1983; Peterson, Fennema, Carpenter, and Loef, in press; Thompson, 1984). These researchers use a variety of methods to identify teachers' conceptions, including interviews, questionnaires, and inferences based on teachers' practices. These studies generally highlight the influence of teachers' assumptions *about* mathematics on their teaching of the subject. Other researchers focus on teachers' *understanding of mathematical concepts and procedures* (e.g., Ball, 1988a; Ball and McDiarmid, in press; Leinhardt and Smith, 1985; Owens, 1987; Post, Behr, Hamel, and Lesh, 1988; Steinberg, Haymore, and Marks, 1985). Using interviews and structured tasks, they explore how teachers think about their mathematical knowledge and how they understand (or *mis*understand) specific ideas. What counts, according to these researchers, is the way teachers organize the field and how they understand and think about concepts (as opposed to just whether they can give "right" answers).

What does it mean to "know" mathematics? Does "knowing math" mean being able do it oneself? Does it mean being able to explain it to someone else? Is subject matter knowledge a question of "knowledge structures"--that is, a function of the richness of the connections among mathematical concepts and principles? What is the relationship among "attitudes," "conceptions," and "knowledge" of mathematics?

Mathematical Understanding: Interweaving Ideas *Of* and *About* the Subject

Understanding mathematics involves a melange of knowledge, beliefs, and feelings about the subject. Substantive knowledge includes propositional and procedural knowledge *of* mathematics--that is, understandings of particular topics (e.g., fractions and trigonometry), procedures (e.g., long division and factoring quadratic equations), and concepts (e.g., quadrilaterals and infinity), and the relationships among these topics, procedures, and concepts. This substantive knowledge of mathematics (Schwab, 1961/1978) is what is most easily recognized by others as "subject matter knowledge."

Another critical dimension, however, is knowledge *about* mathematics.[1] This includes understandings about the nature of knowledge in the discipline--where it comes from, how it changes, and how truth is established. Knowledge about mathematics also includes what it means to "know" and "do" mathematics, the relative centrality of different ideas, as well as what is arbitrary or conventional versus what is necessary or logical, and a sense of the philosophical debates within the discipline. Many of these aspects of mathematics are more often communicated purely by their absence from traditional mathematical study--understanding the history of mathematics, for instance. Rarely do math students learn about the evolution of mathematical ideas or ways of thinking.

Nevertheless, teachers do convey many explicit and implicit messages about the nature of the discipline. If the teacher's guide is the source of right answers, for example, this suggests that the basis for epistemic authority in mathematics does not rest within the knower. Teachers communicate ideas about mathematics in the tasks they give students, from the kinds of uncertainties that emerge in their classes and the ways in which they respond to those uncertainties, as well as from messages about why pupils should learn particular bits of content or study mathematics in general. Finally, in addition to all of this, understanding mathematics is colored by one's emotional responses to the subject and one's inclinations and sense of self in relation to it.

Interviews with prospective and experienced teachers[2] illustrate how understanding mathematics is a product of an interweaving of substantive mathematical knowledge with ideas and feelings about the subject. Asked how she would respond to a student who asked what seven divided by zero is,[3] Laura, a prospective elementary teacher, responded:

> Zero is such a *stupid* number! It's just one of those you wonder why it's there sometimes. I'd just say, "Anything divided by zero is zero. That's just a rule, you just *know* it.". . . You know, it's empty, it's nothing. Anything multiplied by zero is zero. I'd just say, "That's something that you have to learn, you have to know." I think that's how *I* was told. You just *know* it. . . . I'd just say, you know if they were older and they asked me "Why?" I'd just have to start mumbling about something, I don't know. . . . I don't know what. I'd just tell them "Because!" (laughs) That's just the way it is, it's just one of those rules, like in English--sometimes the C sounds like K--you just have to *learn* it. I before E except after C--it's one of those things, in my view.

Laura's answer reveals that she understands division by zero in terms of a rule. She thinks of it as something one must remember, not something one can reason about. Like rules of thumb in English, what to do when one divides by zero is something one just must know. In addition, Laura is impatient about the number zero. She describes it as "stupid": useless and empty. Furthermore, the rule she invokes--"Anything divided by zero is zero"--is also false. In other parts of her interview as well, Laura repeatedly refers to rules that she remembers and some that she has forgotten. She talks about hating math and not being good at it. In this tiny snapshot of Laura's understanding of mathematics, we see that what she does not know in this case is framed by her beliefs about mathematical knowledge and her feelings about its senselessness.

Abby, a prospective secondary teacher, also thought of mathematics in terms of rules and arbitrary facts. Unlike Laura, however, Abby was comfortable with the rules: She could remember them and felt safe within their structure and certainty. When asked about division by zero, she said emphatically:

> I'd just say . . . "It's undefined," and I'd tell them that this is a rule that you should *never* forget that anytime you divide by zero you *can't*. You just can't do it. It's undefined, so . . . you just can't. They should know that anytime you get a number divided by zero, then you did something wrong before. It's just something to remember.

Abby added that dividing by zero is "something that you won't ever be able to do in mathematics, even in calculus." Unlike Laura, Abby's rule was correct--division by zero *is* "undefined"--but, like Laura, her understanding was nested within her larger view of mathematics as a collection of rules to remember. She did not try to make meaning out of the "fact" that division by zero is undefined but simply emphasized that it is not permitted.

Mathematical Understanding: Examining What Teachers Know

Next, in order to illustrate the kind of analysis needed in studying teachers' subject matter knowledge, a closer look will be taken at some prospective elementary and secondary teachers' understanding of place value in multiplying large numbers (Ball, 1988a). Later this topic will be returned to in discussing the role of subject matter knowledge in teaching mathematics. The following discussion is based on the analysis of a single question taken from a series of interviews conducted with teacher education students, half of whom were mathematics majors intending to teach secondary school and half of whom were prospective elementary teachers with no academic major. Analysis of the topic, place value, and of the interview question itself, is followed by a discussion of the results. These results highlight the danger of assuming what teachers understand about the mathematics they teach.

Background: Place Value and Numeration

The question discussed deals with the concept of place value and its role in the algorithm for multiplying large numbers. Some background is necessary to understand the question and the analysis of teachers' responses. The base 10 positional numeration system is part of the working knowledge of most members of our culture. That is, adults read, write, and make sense of written numerals. They know that "56" does not mean 5 + 6 or 11. They know that "04" represents the same quantity as "4" but that "40" does not.

Still, children do not automatically understand this and the way they learn arithmetic may be a hindrance rather than a help to understanding. In fact, some research suggests that place value is particularly difficult for children to learn. Elementary school students may write 365 as 300605, for example, which represents the way the number *sounds* rather than place value. Kamii (1985) argues that traditional math instruction actually forces young children to operate with numerals without understanding what they represent. We have all heard children performing addition calculations reciting, "5 plus 7 is 12, put down the 2, carry the *1*," or doing long division calculations such as 8945 divided by 43 by saying, "43 goes into 89 twice, put up the 2, 2 times 43 is 86" and so on. These "algorithm rhymes"[4] which pupils learn interfere with paying attention to the essence of the numeration system--that numerals have different *values* depending on their *place*. The "1" in the addition rhyme actually means 10. The "89" in the division chant actually means 89 hundred and that "2" represents the fact that there are 2 hundred groups of 43 in 8945.

Place Value in Multiplication Computation

What is the nature of adult working knowledge of place value and numeration? How does it equip teachers to help pupils make sense of written numerals and procedures with numbers? I designed the following question to elicit teachers' understanding of place value in use:

Some eighth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate

```
  123
x 645
-----
```

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

```
  123
x 645
-----
  615
  492
  738
-----
 1845
```

instead of this:

```
    123
  x 645
  -----
    615
   492
  738
  -----
  79335
```

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching eighth grade and you noticed that several of your students were doing this?

Discussion of item. The algorithm for multiplying large numbers is derived from the process of decomposing numbers into "expanded form" and multiplying them in parts. To understand this, one must understand decimal numerals as representations of numbers in terms of hundreds, tens, and ones, that is, in the numeral 123, the 1 represents 1 *hundred*, the 2 represents 2 *tens*, and the 3 represents 3 *ones*. In the following example, 123 x 645, first one multiplies 5 x 123:

```
  123
x   5
-----
  615
```

then 4*0* x 123:

```
  123
x  40
-----
 4920
```

and then 6*00* x 123:

```
  123
x 600
-----
73800
```

In the final step, one adds the results of these three products:

```
  123
x 645
-----
  615
 4920
73800
-----
```

In effect, one is putting the "parts" of the number back together--that is, 645 x 123 = (600 x 123) + (40 x 123) + (5 x 123).

Many people do not write their computation out this way, but rather "shortcut" it by writing:

```
  123
x 645
-----
  615
 492
738
-----
```

This shortcut, in effect, hides the conceptual base of the procedure. Because its logic depends on place value and the distributive property of multiplication over addition, the multiplication algorithm affords a strategic site for investigating the nature of people's mathematical understandings.

Analyzing Teachers' Knowledge of Place Value

In this section the results of this interview question are discussed, focusing on four issues: the interpretation of what people say when they talk about mathematics, the role of explicitness in understanding mathematics for teaching, the connectedness of mathematical understanding, and the interweaving of knowledge of and about mathematics.[5]

Place value or "places"? Some of the prospective teachers' responses were relatively easy to interpret because they focused explicitly either on the role of place value in the algorithm or the steps of the procedure. For example, Mike, an elementary major, said that he would "have to explain about that not being 123 x *4*. That it's 123 x *40*." In contrast, Tara, a prospective elementary teacher focused on the steps:

> I would show them how to line them up correctly. I would do what I still do, which is once I multiply out the first number and then I start to do the second line, put a zero there. That's how I was taught to do it and that's how I still, when I have big numbers to multiply, I do, because otherwise I'd get them too mixed up, probably. It helps to keep everything in line, like after the first line, you do one zero and then you do two zeroes to shift things over.

Mike's answer showed that he understood that "moving the numbers over" is not just a rule to remember, but reflects that 123 x 4 is 4,920, not 492. Tara's understanding was wholly procedural: the numbers must be lined up and the zeroes help you to remember to "shift things over." There was no hint in her answer that she saw any meaningful basis for the procedure.

While these responses were explicit and unambiguous, some prospective teachers' responses were much harder to interpret because they used conceptual language--for example,"the tens place"--to describe procedures, or procedural language--for example, "add a zero"--to (perhaps) refer to concepts. Rhonda's response was an example of this ambiguity:

> You would take the last number and multiply it by all three of the top numbers and you put those underneath and then you start with the next one. You'd want to put it underneath the number that you are using. They aren't understanding that they need to be underneath of that instead of just down in one straight row.

So far this seemed like an answer focused on the rules of the multiplication algorithm. Rhonda was talking only about where to put the numbers and what to do next. But then she said that the students would "probably need to know about places":

> You know, the hundreds, the thousands, you know, whatever. If they don't understand that there is a difference in placing, that could also lead to this if they don't remember. . . . They need to understand that there is a difference in the placing, too.

What did Rhonda mean when she said "placing"? She may have been talking about where to write the numbers--where to *place* them--or she may have been talking about the difference in the *value* of a number depending on its placing. To probe how she understood "places," I asked her why this mattered. She replied:

> Because of the fact that you are working with such a large number, like your second and third numbers are not going to be ones. . . . Your numbers get larger and larger and since you are working with such a large sum, you have to know how to work in the thousands, you know, to keep your numbers that way. I guess it all goes back to them understanding why the numbers should be underneath of what you are multiplying.

She added that she wasn't sure "how that affects the placing."

Rhonda's response was not as clear as Mike's or Tara's. She seemed to focus on lining up the numbers correctly, but then she talked about "places," too. Was her reference to "placing" and "places" conceptual--that is, addressing the *values* of different places within a numeral? Or was Rhonda just talking about "placing" the numbers in the right *place*--so that they would be lined up correctly?

Zero as a "placeholder." Also ambiguous were the responses of several prospective teachers who talked about the importance of writing in zeros in the partial products. Joanie, a secondary candidate and mathematics major, said she would get her students to focus on putting the numbers "in the right places" and would "encourage them to use zero as a placeholder" and Karen, another math major, commented, "We were taught to put a zero here, and a zero there, to represent the places." Chris, an elementary major, tried to explain the role of zero:

> I don't exactly know how to explain it, but something having to do with this first column . . . is the ones, and the next column is the tens, and maybe something like there's a zero, you know, the tens there's always one zero, and so you have--God, I don't know. Like to make it balance out for the tens you'd have to add the zero and for the hundreds you'd have to add two zeros. Something to that effect. *I* don't know.

In some of these cases, interpreting what the prospective teachers understood about place value was difficult. Although their answers focused on how to write the partial products, it is not clear what that meant to them. People could talk about the importance of zero as a "placeholder" and mean simply that using zeros helps one remember to get the numbers lined up correctly.

Some prospective teachers who talked about zeros *did* elaborate their answers very explicitly and their responses reveal different kinds of thinking that can underlie answers focused on "putting down zeros." The responses of Patty and Mike, both elementary teacher candidates, illustrate such differences in thinking. Patty said she would show pupils to "*physically* put a zero every time you moved down a line." She explained that "zero doesn't add anything more to the problem. It's just empty. But instead of having an empty space, you have something to fill in the space so that you can use it as a guideline."

Mike also said he would "make it mandatory that the zeros start showing up" on his pupils' papers. But he explained it differently. He said he would "have to explain about that not being 123 x *4*. That's 123 x 4*0*, which is a multiple of *10*--which has that zero on the right side which is why the zero has to be there."

Both Patty and Mike would have their pupils put the zeros down, yet their explanations revealed strikingly different understandings of the role of zero in our decimal positional numeration system. Patty saw the zero as useful for keeping the columns of numbers lined up but says that zero "adds nothing" to the number. Her statement suggested that she confused "adding zero" *to* a number (78 + 0 = 78) with the role of zero in a *numeral* (e.g., 78*0*). Mike knew that 123 is multiplied by 5, 40, and 600. He said the zeros "have to be there" because the products are "a multiple of 10 off." Still, his response did not show what he understood about the zeros in place value numeration. Was it a rule he had memorized--that multiples of 10 have one zero, multiples of 100 have two zeros, and so forth? Or did he understand why putting a zero "on the right side" produces a number that is ten times the original?

Partial and inexplicit understanding. Those who mentioned "places" and "ones, tens, and hundreds" may have had a partial, fuzzy, understanding of the underlying concepts of place value. Some students figured it out in the course of answering the question. Becky, a post-B.A. student with an undergraduate mathematics major, was one of these. She began her answer much as many others did, focusing on "moving over" from column to column:

> You start in the units column and you multiply that, and then you start in your tens column and so you have to start in your tens column of the next one and you multiply 4 x 123 and then you move over into your hundreds column over here where you're taking 6 x 123.

Then she talked about how she was taught to "put the zeros there because it helped me line up my columns." She pondered this aloud:

> A lot of the time you say, "Well, put a zero here, put a zero there, and zero there, and

> you put a zero here, and a zero there," and you get into the *method* of it and you know that you put a zero here, but they don't really understand *why*. And I think it goes back to the units and tens and hundreds and all that. And that might be an easier way to take a look at it. Cause you're going to take 5 times that, and you take 40, and then 600, and you can see where those zeros come from.

She still wasn't entirely clear about this, though. She said that "when you take the 4 or the 40, you're gonna want to start in, understand that you're working with tens now, so you want to move into the tens column." Becky stopped suddenly and said, "God, I don't know any other way that I'd be able to describe it than, I'd have to think about it." She paused and looked at the problem. Suddenly she realized that 123 x 40 "is going to be the same as this [492] with a zero on it!" She talked to herself under her breath and then a few moments later looked up and said, "Wow, I haven't even thought about it that way before! . . . *that's* where those zeros come from, *oh*! *Wow*, okay."

Although Becky could multiply correctly, she did not know the mathematical principles underlying the procedure. She was, however, able to put different pieces of understanding together and figure it out as she talked. Others who lacked explicit understanding also seemed to realize that there was more to know than just procedures, but could not always uncover the deeper levels. Sarah, an elementary major, struggled and then gave up. Her answer seemed to focus on the rules of lining up the numbers:

> I would explain that every time you move over this isn't ones, this is tens, so it's ten more, so you have to have an extra ten there, you have to put the zero there to hold it in place. Does that make sense?

I asked if it made sense to *her*. She replied, "Oh, I know what I'm saying, I know what I'm thinking, I just, I don't know if I can explain it. . . . I guess it's because the stuff is so basic to me." What Sarah could *say* was that "you have to put the zero there to hold it in place." Moreover, her explanation that in the tens place "it's ten *more*" misrepresents the fact that the value of the tens place is ten *times* more. Still, her comment that she knew what she was thinking but does not know if she can explain it is worth pondering. Sarah seemed to have part of the idea, that something about the value of the places mattered, but was unable to pull it together.

Tacit Versus Explicit Ways of Knowing

Assuming that people have conceptual knowledge of procedures which they have learned to perform is a fallacy (Hatano, 1982). As one of the math majors reflected when he tried to explain the basis for the multiplication procedure, "I absolutely do it [multiplication] by the rote process--I would have to think about it." Certainly many children and adults go through mathematical motions without ever understanding the underlying principles or meaning. For example, while most people can divide fractions using the rule to "invert and multiply," very few are able to connect any meaning to the procedure (Ball, 1988a; NCRTE data, see Ball, in press).[6]

Still, mathematical understanding may also be tacit. Successful mathematicians can unravel perplexing problems without being able to articulate all of what they know. Not unrelated to Schon's (1983) "knowing-in-action," the mathematicians' work reflects both tacit understanding and intuitive and habituated actions. Experts in all domains, while able to perform skillfully, may not always be able to specify the components of or bases for their actions. Their activity nevertheless implies knowledge. Similarly, in everyday life, people understand things which they cannot articulate. For instance, a woman may find her way around the town she grew up in, identifying friends' homes and old hangouts, yet not be able to give directions to a visitor. A man may use colloquial French expressions in speaking French but be unable to explain their meaning to a fellow American.

Polyani (1958) describes what he calls the "ineffable domain"--those things about which our tacit understanding far exceeds our capacity to articulate what we know. He argues further that "nothing we know can be said precisely, and so what I call `ineffable' may simply mean something I know and can describe even less precisely than usual, or even only very vaguely" (p. 88). It is unclear whether we would want to say that the woman understands her way around less well than someone who can give directions, or that the man understands French less well than someone who can translate. Clumsy attempts to articulate understanding may reflect an area in which, according to Polyani (1958) the tacit predominates.

In contrast, apparent clumsiness in expression may not be clumsy or inarticulate at all, but rather may reflect how the speaker actually understands what he or she is talking about. Orr (1987) argues that teachers often "fill in" the gaps in what their pupils say, assuming they know what the pupils "mean." She said that when her high school geometry students would talk about distances as locations and locations as distances, she thought these were careless mistakes or awkward wording. Suddenly it occurred to her that these nonstandard ways of talking might actually represent nonstandard understandings of the relationship between location and distance. She began asking some different questions of her students to try to elicit what they understood--asking them to construct diagrams showing where certain cities were located and the distances among them, for example. She discovered in case after case that her students' explanations were accurate reflections of how they were thinking.

Just like mathematicians, ordinary people do things in mathematics--and do them correctly--which they cannot, however, explain. The prospective teachers whom I interviewed all clearly knew the steps of the traditional procedure for multiplying large numbers and could calculate the answer correctly. Yet very few had examined this habituated procedure. Almost no one was able to talk about *why* the numbers "move over" in the partial products, except to say that the product of 123 x

4 must be "lined up under the 4 because that's what you're multiplying by." This raises two issues critical for research on teacher knowledge: one methodological and one theoretical.

What do teachers understand? Problems of inference. Analyzing teachers' knowledge is complicated by the extent to which they are able to talk or otherwise represent that knowledge. If someone talks about "lining up the numbers," one cannot fairly assume that the person has no understanding of the role of place value in the multiplication algorithm. At a tacit level, the person may understand that 123 x 4 is really 123 x 4*0*, but may never explicitly consider this in performing or thinking about the procedure. This issue clearly presents methodological problems of inference in studies of teachers' subject matter knowledge. A second consideration, however, affords a way out of this methodological tangle.

What do teachers need to know? Tacit knowledge, whatever its role in independent mathematical activity, is inadequate for teaching. In order to help someone else understand and do mathematics, being able to "do it" oneself is not sufficient. A necessary level of knowledge for teaching involves being able to talk about mathematics, not just describing the steps for following an algorithm, but also about the judgments made and the meanings and reasons for certain relationships or procedures.[7] Explicit knowledge of mathematics entails more than saying the words of mathematical statements or formulas; rather, it must include language that goes beyond the surface mathematical representation. Explicit knowledge involves reasons and relationships: being able to explain *why*, as well as being able to relate particular ideas or procedures to others within mathematics. This is more than "metacognitive awareness" of the processes used in solving a mathematics problem or carrying out a procedure; it includes the ability to talk about and model concepts and procedures.[8]

The Degree of Connectedness Within Teachers' Substantive Knowledge of Mathematics

To investigate the degree to which teachers' knowledge of place value is connected across contexts, the elementary teacher candidates' understanding of place value was explored in a structured exercise focused on teaching subtraction with regrouping (another procedure dependent on concepts of place value). In this exercise, which was longer than the rest, the teacher candidates were asked to examine a section from a second-grade math book. This section (two pages) dealt with subtracting two-digit numbers with regrouping. The teacher candidates were asked to appraise the section, to talk about what they perceived as its strengths and weaknesses, and then to describe how they might go about helping second graders to learn "this." I did not specify what "this" was because I wanted to see what they would focus on. I also asked them what they thought pupils would need to know before they could learn this, and what they would use as evidence that their pupils were "getting it." Finally they examined an actual second grader's work on one of the pages, and were asked to talk about what they thought she understood and what they would do next with her.

In their responses, almost all of the teacher candidates focused explicitly on concepts of place value. Their responses showed that they were aware that "tens and ones" played some sort of role in teaching subtraction with regrouping (which they all referred to as "borrowing").[9] For some, this awareness of tens and ones was at the surface, readily accessible. For example, Tara described what she would say:

> I would say, you know, obviously these numbers, you can't subtract in your head. Alright, you have to cross out one of the tens from the top. And put it over in the ones column on the top, so you are able to subtract the two numbers. And then when you cross that tens number, change it, like subtract 1 from it. So you change, like if it was 64, change it to uh, you know, the 6 to a 5, and the 4 to a 14. And maybe I would show them, like 64, like maybe I would write 64 on the board. And then put that it equals 50 plus 14, so they see it is still the same amount.

$$\begin{array}{r} \mathbf{5\ 14} \\ \not{\mathbf{6}}\ \ \not{\mathbf{4}} \\ \underline{\mathbf{-\ 4\ \ 6}} \end{array}$$

Tara, in the midst of a procedural description ("change the 6 to a 5"), explicitly added an important piece of conceptual understanding: that 64 equals 50 plus 14 and so the crossing out has not *changed* the value of the number.

Almost all the teacher candidates were more explicit about place value when talking about subtraction with regrouping than they were when they discussed the multiplication algorithm. With multiplication, for instance, Tara focused on "lining up the numbers" and "shifting things over" on each line. She did not seem to understand that the partial product written as 492 was really 4920 ("adding the zero just keeps everything in line"). Yet, in talking about subtraction with regrouping, Tara talked explicitly about 50 + 14 being "the same amount" as 64.

The teacher candidates seemed to understand the role of "tens and ones," or place value, in "borrowing" but did not connect that understanding with the multiplication algorithm. Their understanding of place value was compartmentalized within specific contexts (e.g., borrowing), and not readily accessible in other relevant ones (e.g., multiplication computation). Similar evidence of fragmented understandings emerged within other topics examined in the interviews--division, for example. Prospective teachers did not connect the concept of division across different division contexts: division of fractions, division by zero, and division in algebra. Instead, they treated each as a specific case, for which they had to invoke a particular rule or procedure.[10] These results point to the importance of investigating the connections within teachers' substantive understanding of mathematics. In seeking to examine what teachers know, researchers should create opportunities to explore teachers' knowledge of particular concepts across different contexts or from a variety of perspectives.

The Interaction of Knowledge *Of* and *About* Mathematics

In addition to the explicitness and connectedness of teachers' knowledge of concepts and procedures, another critical area of inquiry and analysis is the way in which their ideas *about* mathematics influence their representations *of* mathematics. What do they emphasize? What stands out to them about the mathematical issues they confront?

The prospective teachers tended to focus on the procedures of multiplication for reasons that also went beyond the nature of their substantive understanding of place value and had more to do with their ideas *about* mathematics. Some of the predominant assumptions included the following:

- Doing mathematics means following set procedures step-by-step to arrive at answers.
- Knowing mathematics means knowing "how to do it."
- Mathematics is largely an arbitrary collection of facts and rules.
- A primary reason to learn mathematics is to progress to the next level in school.
- Another main purpose for learning math is to be able to calculate prices at the store.
- Most mathematical ideas have little or no relationship to real objects and therefore can be represented only symbolically.

The prospective teachers' assumptions about the nature of mathematical knowledge and what it means to know something in mathematics formed the boundaries of what they considered to be a response on all of the interview questions. In talking about the multiplication question, for example, one commented that "you just have to move the number over a place value every time--it's just knowing how to do something." Several others vowed that they would "enforce" or "make mandatory" that pupils use "placeholders" in order to remember to move the numbers over on each row. No one suggested using objects, pictures, or real situations to model the procedure.

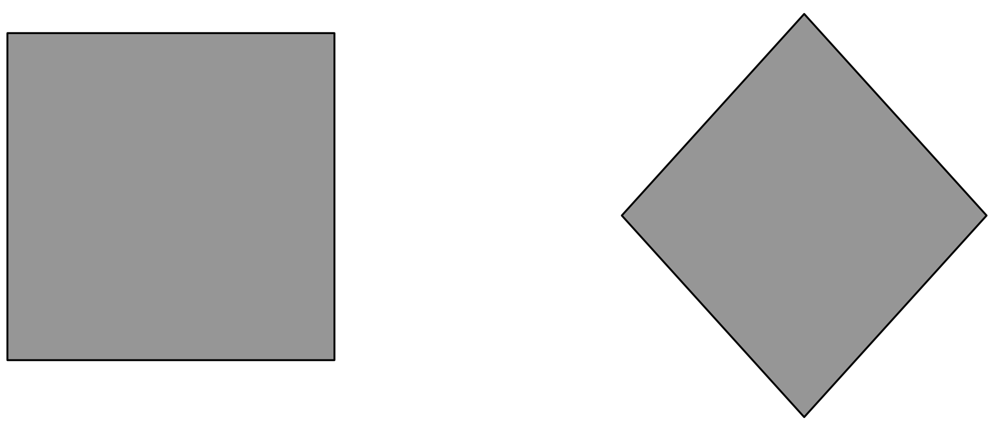
Obviously the prospective teachers' ideas about mathematics do not exist separately from their substantive understandings of particular concepts or procedures. Most of them did not have access to any explicit understanding of why the multiplication algorithm works. As such, they could do nothing else but respond in terms of rules and procedures. Still, at the same time, many were emphatic about the importance of teaching students to follow the steps correctly and they tried to think of ways to "imbed" those steps into students' heads, rather than seeking to figure out the underlying ideas.

Although people have many ideas about the nature of mathematics, these ideas are generally implicit, built up out of years of experience in math classrooms and from living in a culture in which mathematics is both revered and reviled. While such ideas influenced the ways in which they experienced mathematics, the prospective teachers seemed to take their assumptions about mathematics for granted. Unlike their understandings of the *substance* of mathematics, which some of them wished to increase or deepen, the teacher candidates did not focus on their understandings about mathematics. They did not seem dissatisfied with them, nor did they even seem to think explicitly about them.

The Role of Subject Matter Knowledge in Teaching Mathematics

The discussion thus far shows that seriously examining and analyzing teachers' knowledge of mathematics is a complicated endeavor. Another knot in the pursuit of understanding the role of subject matter knowledge in teaching mathematics, however, lies in the nonlinear relationship between knowledge of mathematics and teaching. In teaching, teachers' understandings and beliefs about mathematics *interact with* their ideas about the teaching and learning of mathematics and their ideas about pupils, teachers, and the context of classrooms.

To make this assertion more concrete, put yourself in a math teacher's shoes. Imagine you are teaching first grade and you are teaching your students to identify geometric shapes. One little girl points to the blue wooden square and says that it's a rectangle. Then another child tilts the square and says that now it is a diamond.



How would you respond? Or suppose you are a high school math teacher. Bored and frustrated, the students in your fifth-period geometry class demand to know why they have to learn proofs. What would you say? Before reading further, take a moment to consider what you *would* do or say next.

Now consider your responses. The choices you made in each of these situations was based on what you interpreted it to be *about*. In other words, teaching is as much a process of *problem setting* as it one of *problem solving*. Your interpretation of each situation, perhaps implicit, was shaped by two main factors: your knowledge about pupils of different ages and your understanding of the mathematics involved. For instance, did you think that either of the children in the first example had said something either incorrect or insightful? *Can* a square be correctly labeled a rectangle or a diamond? Is a diamond a mathematical term? What is the effect of changing the orientation of a geometric shape? How does one answer such questions in mathematics? Are these issues things that first graders can or need to understand? Would exploring the hierarchical relationship between rectangles and squares be confusing for the rest of the class?

In the second example, why did you think these high school students were bored and frustrated? Maybe you hated proofs in high school, too, and you sympathized with them. Maybe you thought the pupils were just trying to get you off on a tangent. Why *do* you think you are teaching proofs? These questions, and others like them, played a role in the way you interpreted and defined each situation.

Having framed each situation, your response--what you think you would *do*--was then nested within your assumptions about good math teaching. These assumptions are grounded in your ideas about how pupils of particular ages learn, what you believe about the teacher's role, what you think is important to learn in math and what you know about the school mathematics curriculum, as well as your ideas about the context of classroom learning. Your own understanding of mathematics is a critical factor in this interplay of interpretation and response in teaching mathematics.

The examples illustrate that the role of subject matter knowledge in teaching can be more complicated than whether or not you can define a rectangle. But, what matters about your knowledge of mathematics depends on a host of other factors which, taken together, comprise your view of mathematics teaching. It is a cycle: What you need to understand about shapes or proofs depends on what you think the point of teaching geometry is, which is in turn connected to your larger understanding of mathematics in general, and geometry in particular. Unanimity about good math teaching does not exist among mathematics educators, researchers, or teachers; to gloss over such differences of view is to doom current research efforts to a new set of failures.

To establish this argument, three cases of the teaching of long multiplication in fourth grade are presented. The three teachers are all teaching the same topic, to the same age students, and, in the vignettes, are at similar points in their work on this topic. However, because of what these three teachers understand about mathematics, what they believe about teaching and learning mathematics, about pupils, and about the context of classroom teaching, they approach the teaching of multiplication in distinctly different ways. The purpose of examining these cases is to highlight the role of subject matter knowledge in teaching mathematics, showing the ways in which subject matter knowledge interacts with other kinds of knowledge in teaching mathematics. After each case, each teacher's approach and the factors that seem to shape that approach are summarized. Following the three cases is an analytic discussion of the three cases and what they demonstrate about the interaction of subject matter knowledge in teaching mathematics.

Bridget Smith[11]

Bridget Smith teaches in a small suburban community of white middle class families. She has been teaching for over 20 years. Her preferred approach to teaching math is to "individualize"--allowing the pupils to proceed at their own pace through the text material and, when necessary, "going over and reteaching with them skills that they either had forgotten or had never been taught." Sometimes she works with the whole group if she wants to "expose all of them" to something they haven't yet done--such as long division or multiplying by two numbers--or if the pupils are unable to work well alone, which is the case with this year's group.

Smith believes that some of her pupils are naturally good at math while others have personalities that make it difficult for them to comprehend mathematics at all. She describes one of her best students:

> He is capable of listening to a direction and following it and catching on very quickly. He has got, he has just got real good math sense. Very bright boy . . . he just has a real uncanny sense of just listening and it all makes sense to him. It just makes sense.

She thinks "it is just something about him" that enables him to be successful in doing math. In contrast, her struggling students

> always have to be reminded that they have to borrow in subtracting. They know how to do it but to give them a problem if they have not been working in subtracting, they just take the smallest number away from the largest number and cannot understand why it is wrong. They have to be reminded to move, if they multiply by a second number, to go over one place. And in dividing by two numbers, they just cannot handle that at all.

Smith believes that if researchers could figure out "what kinds of personalities are like that," she might better be able to help these kinds of pupils "catch on."

Smith's goal in teaching math is to teach for mastery of the procedures required in fourth grade: adding, subtracting, multiplying, and dividing. She explains, "What I am after is the answer." This is important to her because the pupils' computational skills determine their placements into groups in fifth grade. She finds that, for many of her students, it is just a matter of remembering what to do. For example, on Mondays, they often have forgotten the steps of the procedures and need to be reminded.

Then their papers are better on Tuesday.

To help them remember, Smith gives her students mnemonic aids. For example, to remind them what to do in long division, she wrote the following on the board:

÷
x
-
bring down

To check division:
x ans by number outside
and + remainder R

This mnemonic represented the steps of the division algorithm--"divide, multiply, subtract, bring down" and the process by which division solutions can be "checked."

I presented Smith with the place value in multiplication question discussed in the last section. It was a familiar problem to her since she teaches multiplication. She said she would try to

> get the students to see that when they multiply, well, 3 x 5, and that would come under the 5. And then because we have used that, that becomes a zero and they could hold that place with a zero if they want to. And then when they multiply 3 x 4, it would come in the same column as the 4. And when you multiply 3 times 6, it is in the same column as the 6.

```
   123
x  645
   615
  492
 738
 79335
```

She explained that, although the second partial product *is* 4,92*0* and that is why the zero makes sense, she gives her pupils "the option" about putting the zeros in because some students get "very confused about the zero." Although Smith knows why the 492 is shifted over (i.e., because it is really 4,920), she said she does not talk with her students about the meaning of the procedure. Her concern is to get the students to be able to perform the computation and so she emphasizes using the zero only to help them "hold the place" and to remember to move one column over on each line.

Teaching multiplication. Smith's approach to helping her students learn to multiply large numbers is to provide reminders, practice, and feedback. One day she starts class by distributing a ditto with multiplication problems. She has been working on multiplication with her class for a while, but does not think they have entirely mastered it. Without comment or question, the 20 students begin the computations; the room is absolutely silent. Smith has written a reminder on the board:

x
x
+

signifying the steps for multiplying large numbers--multiply, multiply, then add. Smith walks around, looking over pupils' shoulders at their work. She pats one of the girls on the shoulder, smiles, and comments quietly, "Good job!" She circulates to other students, placing her hand on their shoulders as she pauses to glance at their papers.

After about 15 minutes everyone has completed the ditto and they have turned them into the basket of finished assignments. Smith announces that the class will now go over the problems together on the board so that "you can see if you were on the right track." She writes the first five problems on the board and calls five pupils up to do them in front of the class. Since they have turned their papers in, they are doing these from scratch, not copying their own work. When they are done, Smith checks each one against her answer sheet, and marks a large C above each one that is correct. Discovering two that are incorrect, she clucks: "Uh-uh-uh-uh-uh! We're doing *multiplication* here." She has those two children return to the board to redo their problems.

By now, the noise level is high and few children are looking at the board. Smith erases the first five, and writes three more problems on the board. She calls students to come up to do them, including the "bonus problem," a 3-digit by 3-digit multiplication, something she said they had not yet been taught. She picks Jon, her best student, to do the bonus problem.

$$\begin{array}{r} 223 \\ \underline{\times\ 417} \end{array}$$

Smith checks over the students' work on the board and then asks, "Anybody have any questions about the more difficult problems? How many of you feel you had the bonus question right?" She walks them through it: "How many of you remember--when you multiply by the 7, you put it here [under the 7], when you multiply by the 1, you put a zero? Then when you multiply by *another* number, you put *two* zeroes here." She asks again if anyone has any questions. No one does. She reminds them of their social studies homework for tomorrow and the class is dismissed.

Smith's approach. Ms. Smith thinks it is most important for students to become proficient in the multiplication algorithm, to be able to put together the steps in order to produce the right answer. Her eye is fixed on her students' future in school, that is, on their placement level in middle school. What she emphasizes derives from this concentration. Although she understands the conceptual basis for the rule to "shift the numbers over," she does not feel this is important for her students to understand: No one will expect them to know that in fifth grade (or ever). Her assessment of her students' ability also reflects her conception of mathematics as a set of rules to remember and follow: The better students listen well and follow directions; the weaker ones have to be reminded all the time.

Overall, correct answers are Smith's goal and she is the source of these answers in her class. She checks the students' written work and grades it. When she has pupils do problems in front of the class, she herself marks them right or wrong. Smith's purpose in going over the assigned problems is so that students can see the right answers, not to discuss the reasonableness of the answers or the process by which they were obtained. When two students got wrong answers at the board, Smith had them redo the problems; she did not ask the others to try to figure out where the errors lay.

Interestingly, she did offer them a problem which they have not been taught to do (a 3-digit times 3-digit multiplication). In keeping with her belief that math just makes sense to some students, though, she picks her best student to do this one at the board. When he gets it right, she does not choose to engage the class in a discussion of what he did or why it made sense. Instead, she tells them the steps of the procedure and asks if anyone has any questions.

Finally, Smith's approach is based on her belief that students learn mathematics by independently practicing examples in a quiet setting until they remember the steps. Her role is to give them structured opportunities to practice, provides them with helpful mnemonic aids to reduce their tendency to forget parts of the procedures, and confirms the accuracy of their work. The next teacher seems in some ways to take a similar approach, yet some significant differences in emphasis and rationale are apparent, reflecting a different interaction among subject matter knowledge, ideas about teaching and learning, about learners, and about the context of the classroom.

Belinda Rosen[12]

A teacher for over 10 years, Belinda Rosen teaches in a white middle class suburban community. Her school regroups children across classes for mathematics and reading instruction, and Belinda receives the weakest students in the fourth grade. She uses a whole group approach to teaching math and focuses heavily on computational skills. She is, however, very torn about the appropriateness of this focus, wondering how much time she should spend on computational skills on one hand and problem solving on the other. She realizes that "math is not just computation and the books are written as if math were just computation." This pulls her to do "a variety of things," such as "time, money, graphs, and Cuisenaire rods." At the same time, she acknowledges that her pupils will "have to be able to subtract if they are going to have a checkbook" or buy wallpaper. In addition, they must know how to add, subtract, multiply, and divide for fifth grade.

Rosen's goals are shaped by her ideas about her pupils. Because they are weak, she believes she should emphasize following directions and understanding math vocabulary: "To really get that clear when you say *product*, what does that mean, what does that word key, you know?--That it should be multiplication." She said she tries to "inculcate" them with some of the essential material, so that "when and if something clicks," they'll have had exposure to it before. Rosen also wants the students to develop more confidence that they can figure things out for themselves and to enjoy math class. She thinks variety is important just to help her pupils feel happy about coming to class, and she gives little rewards to encourage them.

Rosen thinks that some students are perhaps "math disabled." They may have "great reasoning ability," but they cannot remember what they have been taught from one day to the next: Every day is "a *brand* new day." While she cares about finding ways to help these students and believes she can "get to them *eventually*," she thinks some approaches would not work with her class because they would not be able to handle them. "Discipline" is her least favorite part of teaching, and she thinks that these weaker students tend to be more distractable and have more behavior problems.

When I presented Rosen with the place value in multiplication question she thought that teaching the students to "do a placeholder" would help. She said she would emphasize the sequence of steps and show them that "the first line down is one placeholder, second one down is two placeholders." Her strategy would be to do several problems together with them, starting with easy ones that "didn't have regrouping or hard math facts" so that she could emphasize "the process."

Rosen also suggested a couple of other strategies that she thought would help. She said she has the students put an asterisk inside the placeholder (the zero) so that they don't "get confused with other zeros":

> If the first one that they had to multiply in that row was, say, 5 x 8, and they are going to put down another zero, I don't want them to get confused about whether they had actually already put their placeholder down.

```
 375
x 83
1125
  0
```

She said that, with her students, she would also emphasize writing neatly because poor penmanship is often the root of errors in lining the numbers up correctly. She said she sometimes uses graph paper to help them keep the number lined up.

Teaching multiplication. Rosen has been working on multiplication with her pupils for several days. One day she asks a pupil to distribute chalk to the others, all of whom have individual slates at their desks, and says that they are going to work on 3-digit times 2-digit multiplication. She announces that she will give a sticker to everyone who works on multiplication if she doesn't have to talk to them about their behavior. The following series illustrates the detail with which Rosen proceeded to "work on" multiplication with her class. She writes the following on the board and asks what the first step is:

```
 243
x 22
```

Ronnie:	3 x 2.
Rosen:	We're going to take the number that's in the ones column and we're going to multiply it, and 3 x 2, Karen?
Karen:	6
Rosen:	And what's our next step?
S:	2 x 4
Rosen:	2 x 4. And Darrell, what are you going to say for 2 x 4?
Darrell:	(pause, being silly with a silly voice) 4
Rosen:	4? 1 x 4 is 4. What's 2 x 4?
Darrell:	2 x 4? . . . 8
	(Several other students applaud.)
Rosen:	And then?
S:	2 x 2
Rosen:	And what's our next step?
S:	Put the placeholder down.

Rosen repeats, "Put the placeholder down," and writes a zero with an asterisk inside under the ones column in the second row.

243
x 22
486
0

Rosen: We're going to multiply the number in the tens column by 3.
S: 6

The pupils and teacher continue in this manner until they have finished multiplying. She reminds them to put a plus sign down.

243
x 22
486
+ 4860

Rosen: 6 + 0? Karen?
Karen: 6
Rosen: And then?
S: 14
(Rosen writes the 4 down and carries the 1.)
Rosen: Darrell, take over?
Darrell: 8 x 4 . . .
Rosen breaks in: "We're going to *add*, remember, Darrell? Remember about *adding*?"
Darrell: 8 + 4 is 12.

They finish the problem together. Rosen says she is going to give each of them a problem, "but before I do, let's review the steps. What's our first step?"

S: Multiply?
Rosen: We're going to multiply by the number in the ones place. (She writes "mult x #1's place.") Jim, what's our next step?
Jim: Placeholder.
Rosen: Then we're going to do placeholder. (She writes "placeholder.") What are we going to do next?
S: Multiply.
Rosen: We're going to multiply by the number in the tens place.
(She writes "mult x #10 place.")
Rosen: Next step? Lynn?
Lynn: Add
(Rosen writes "add.")

1. Mult x #1's place
2. Placeholder

3. mult x #10's place
4. Add

The teacher writes another problem on the board and the children proceed to do it on their slates. She walks around helping kids, mostly reminding them about the placeholder step and urging them to work slowly and carefully. Rosen talks one girl through adding up the products. When everyone is done, Rosen goes through the problem, step by step, on the board. She gives the pupils a few more problems to do; the last one requires regrouping (none of the others have). Before class ends, they go over this last one together. Rosen walks them through the steps, asking different pupils to calculate each step as before. Rosen passes out a ditto with more multiplication problems and assigns the first two rows for seatwork.

Rosen's approach. Ms. Rosen is driven by her concern for her weak pupils and her ideas about what they need. She wants them to be successful in the school curriculum, but knows that this has been very difficult for them. She tries to offer them as much support as possible to enable them to do multiplication correctly. She not only spells out the steps in detail and reviews them several times, she also carefully walks the class through many problems together.

Rosen is aware of many little things that go wrong in her pupils' use of the algorithm--such as forgetting whether the zero one has written down is a "placeholder" or part of the next computational step--and she tries to build in safeguards--such as the asterisk--to ensure that pupils will not fall into these traps. One of these pitfalls is that students forget to *add* the partial products and multiply them instead. Rosen tells the students to put an asterisk inside the placeholder zero and to write down a plus sign in order to help them remember what to do. For Rosen, learning the steps is what there *is* to know about multiplication. The help she provides is designed to enable these students, who have trouble learning math, to be successful.

The students are kept active in Rosen's class--with paper and pencil tasks, with slates, or by being called upon to provide answers--because she thinks they are very distractable and that she must keep their attention in order to help them learn. She even offers stickers to encourage them to stay on task. On some days, Rosen provides a break from computation by doing "time, money, graphs, or Cuisenaire rods." This list reflects a conception of worthwhile mathematics curriculum shaped by beliefs about pupils, grounded more in utility and fun than in mathematical significance. Time and money are mathematical topics in school only; Cuisenaire rods are a representational tool, not content.

Rosen is the one with the answers, the source of validation in her classroom. She leads them skillfully through the steps ("And what do we do next? What's our next step?") When they say something wrong--for example, when Darrell says that 2 x 4 is 4--she corrects them. She does not wait to see if other students object or if the student who has made the error catches it. Rosen says she wants her students to be able to "figure things out for themselves"; what she means is that she wants them to be able to follow the procedures without guidance. Although she might not argue with such a goal, she is not focusing on developing conceptual autonomy or epistemic power.

The third teacher approaches the teaching of multiplication in an entirely different way than either Smith or Rosen. Her approach, driven by a view of mathematics as a discipline, reflects a different pattern of interaction among subject matter, teaching and learning, learners, and the classroom context.

Magdalene Lampert[13]

Magdalene Lampert teaches fourth-grade mathematics in a heterogeneous school in which over a third of the students speak English as a second language. An experienced elementary teacher of over 10 years, Lampert is also a university professor and researcher, who draws on her classroom teaching in her research and writing. Her students' mathematical skills range broadly, from those who do not add or subtract accurately to those who can add, subtract, multiply, and divide with whole numbers. In Lampert's approach to teaching mathematics, learning what mathematics is and how one engages in it are goals purposefully coequal and interconnected with acquiring the "stuff"--concepts and procedures--of mathematics. Lampert's goal is to help students acquire the mathematical skills and understanding necessary to judge the validity of their own ideas and results, in other words, to be "independent learners" of mathematics (or to be "empowered"; see Prawat, 1988).

Lampert's pedagogy subtly blends goal and process. For example, when students give answers or make assertions, Lampert almost always comes back with, "Why do you think that?" or "How did you figure that out?" She explains that this strategy helps her to understand how her students are thinking, critical information for subsequent pedagogical decisions. Yet she also uses this strategy because it contributes to her goal of fostering "a habit of discourse in the classroom in which work in mathematics is referred back to the knower to answer questions of reasonability" (Lampert, 1986, p. 317).

In her teaching, Lampert tries to balance her pedagogical responsibility to make sure students learn what they are supposed to know with her commitment to helping students invent and construct mathematical ideas and procedures. She, for example, chooses the tasks on which students work. Their solutions, however, form the basis for the class discussion and further work. Lampert also introduces various representational systems, such as coins and the number line, with which students can explore mathematical problems. She models mathematical thinking and activity, and asks questions that push students to examine and articulate their ideas.

Although she leads class discussion, its substance grows out of students' ideas and proposals for strategies. Perhaps most critical in this approach is her role in guiding the direction, balance, and rhythm of classroom discourse by deciding which points the group should pursue, which questions to play down, which issues to table for the moment. This leads to inevitable dilemmas about when and how much to intervene in their puzzlements. For example, she describes an occasion in her class when a heated debate arose about whether decimal numbers are actually negative. She pondered what she should do:

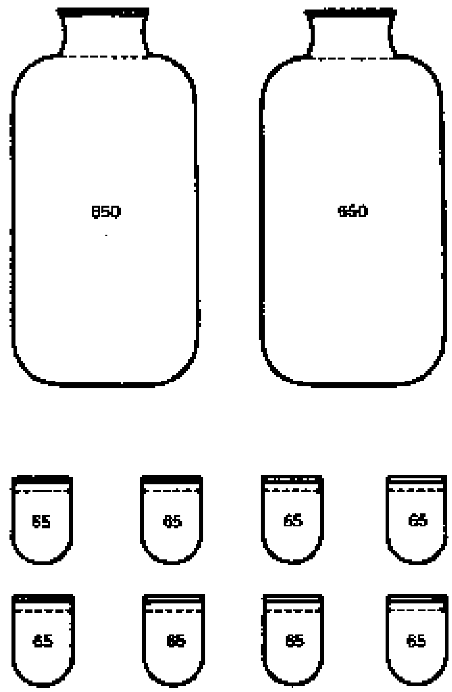
> Should my first priority in the second lesson be simply to tell these students that decimals are definitely *not* negative numbers? My wish to present mathematics as a subject in which legitimate conclusions are based on reasoning, rather than acquiescing to teacherly authority led me away from this approach. (Lampert, in press, p. 24)

Lampert believes that all her students are capable of learning and engaging in significant mathematics and she corroborates that conviction frequently, noting with pleasure when her pupils become embroiled with the meaning of negative numbers or the infinity of numbers between zero and one (Lampert, in press). She also assumes that elementary school students can be absorbed by abstract work as well as by problems centered in interesting real-life contexts. Sometimes she constructs problems that draw on familiar knowledge, such as money, and at other times sets tasks which are wholly separate from her pupils' everyday experience.

Lampert, in preparing to teach her students to multiply large numbers, analyzed what it means to understand multiplication. Knowledge of multiplication, she decided, could be of four kinds: intuitive, computational, principled, or concrete (Lampert, 1986). Intuitive knowledge of multiplication is reflected in people's informal reasoning in solving real-world multiplication problems, independent of formal knowledge about multiplication. Computational knowledge refers to the traditional procedural knowledge taught in school; principled knowledge is the *why* of computational knowledge--for example, knowing that 23 x 5 can be calculated by decomposing 23 into 20 + 3, multiplying 20 x 5 and 3 x 5, and adding the resulting products. Concrete knowledge is being able to represent a problem with objects in order to solve it. Based on her analysis of the content, Lampert determined that what she wanted was to provide experiences that would enable her students to strengthen their competence in each of these four ways of knowing about multiplication and to help them to build connections among them (p. 314).

Teaching multiplication. In Lampert's (1986) series of lessons on multiplication she reached a point comparable with Smith and Rosen--midway in helping students learn about multiplication. She had decided to engage her students in telling and illustrating multiplication stories. After a couple of lessons in which the class constructed stories and pictures to represent multiplications like 12 x 4, she introduced 2-digit by 2-digit problems. By now the pupils were familiar with this mode of representation and she felt they were ready to take on this more complex challenge.

Lampert asks her pupils to come up with a story for 28 x 65. Colleen suggests 28 glasses with 65 drops of water in each glass. Lampert accepts this proposal, but says she does not want to draw 28 glasses on the board so she will draw big jugs that hold the equivalent of 10 glasses. She asks the class how many jugs and how many glasses she needs in order to represent Colleen's 28 glasses. They tell her: 2 jugs and 8 glasses. As she draws big jugs and glasses on the chalkboard, she queries again: How many drops of water in each glass and in each jug? Once again, students reply. Each time a student answers, Lampert asks the student to explain his or her answer. The chalkboard drawing looks like this now:



Next Lampert asks the class how they can find out how many drops of water there are altogether. They say that they should add the jugs and glasses together. The pupils understand readily that the two jugs contain a total of 1,300 drops. Lampert then proceeds to teach them a "trick" that makes it easier to add the 8 glasses together: She suggests that they could take 5 drops of water out of each glass and put them in another container, leaving 60 drops in each glass. She asks the class how many drops would there be in all the glasses then. Someone explains that it would be 480 with just 40 drops in the other container. Combining those yields 520 drops, and adding those to the 1,300 equals 1,820. Lampert points out that by using "clever groupings" they have figured out 28 x 65 without doing any paper-pencil computation. Just as she thinks they have finished this problem and is ready to move on, however, one of her girls, Ko, says she has come up with "another way of thinking about it." Lampert, listening intently, writes Ko's explanation on the board "so as to give it equal weight in the eyes of the class" (p. 329).

Ko proposes that they could have thought about three jugs. Two jugs would hold 1,300 drops, but the third would have 2 glasses, or 130 drops, too much water. She explains that if you remove the 130 drops from the third jug, you are left with 520 drops (650-130), which, added to the other two jugs yields a total of 1,820 drops of water. Lampert draws a picture of Ko's idea on the board and together the class explores why it made sense mathematically.

Lampert spent a few more days using students' stories to draw pictures and examine the ways in which the numbers could be decomposed, multiplied in parts, and recombined. Next she constructed assignments which required the students to make up and illustrate stories, as well as write the numerical representations. Sometimes she asked them to decompose and recombine the quantities in more than one way. They presented and defended their solutions to other class members. Lampert moved on from this to work with her class on the meanings of the steps in paper-pencil computation, using alternative algorithms (i.e., "no-carry" method) as well as the traditional one.

In writing about this work, Lampert (1986) reflected on the contributions of this series of lessons to her overall goals in teaching multiplication:

> They were using the language and drawings we had practiced to build a bridge between their intuitive knowledge about how concrete knowledge can be grouped for counting and the meaning of arithmetic procedures using arithmetic symbols. By rewarding them for inventing reasonable procedures rather than for simply finding the correct answer, I was able to communicate a broader view of what it means to know mathematics and learn something from what they were doing about how they would use mathematical procedures in a concrete context. (p. 330)

She observed also that her students were gaining in their ability to substantiate their claims using reasons "that came very close to the steps of a mathematical proof as well as inventing "legitimate variations on both concrete and computational procedures" (Lampert, 1986, p. 337).

Lampert's approach. Lampert draws the strategy and rationale for her approach from the discipline of mathematics itself: The goal is to help students develop mathematical power and to become active participants in mathematics as a system of human thought. To do this, pupils must learn to make sense of and use concepts and procedures that others have invented--the "body" of accumulated knowledge in the discipline--but they also must have experience in developing and pursuing mathematical hunches themselves, inventing mathematics, and learning to make mathematical arguments for their ideas. Good mathematics teaching, according to this perspective, should result in meaningful understandings of concepts and procedures, but also in explicit and appropriate understandings *about* mathematics: what it means to "do" mathematics and how one establishes the validity of answers, for instance.

While exploring the mathematical foundations of multiplication, Lampert's students were also encountering some strong and intentional messages about what it means to "do" and to "know" mathematics. Lampert consciously tried to ensure that students would have to turn back upon themselves and upon their mathematical knowledge in order to validate their answers and strategies. She explains that the essence of her approach is to teach her pupils

> to use representational tools to reason about numerical relationships. In the public discourse of the classroom, such reasoning occurs as argument among peers and between students and teacher. *It is the ability to participate in such arguments that is the mark of mathematics learning* [emphasis added]. (Lampert, in press)

Lampert's approach thereby fuses assumptions about how learning occurs with a view of what it means to do and to know mathematics. Both entail and depend upon discussion and argument, pursued within a community of shared standards and interests. In the interest of learning, Lampert strives to create a classroom culture in which this kind of intellectual activity is the norm (different from the traditional context of classroom life); within this culture she simultaneously constructs an explicit curriculum of mathematical activity.

Smith, Rosen, and Lampert: What is Mathematics?

Whether they do it consciously or not, teachers represent the subject to students through their teaching. With the tasks that they select, the explanations that they provide, and the kinds of things that they emphasize, teachers convey messages to their student about both the *substance* and the *nature* of mathematical knowledge (McDiarmid, Ball, and Anderson, in press).

Looking at substance first, how do Smith, Rosen, and Lampert represent multiplication to their students? Smith and Rosen use the symbolic form only, without connection to concrete or real-world objects. Neither do they use visual representations. Multiplication is represented as symbolic manipulation and shorthand language is provided to summarize the procedure so that students will remember the steps and their order.

Lampert represents multiplication using drawings. While the objects (containers and water drops) were proposed by a student, Lampert chose the specific pictorial representation of the student's idea to represent an essential conceptual component of the procedure: grouping by tens. Instead of drawing 28 glasses, each with 65 drops of water, she feigned laziness and suggested drawing jugs that hold the equivalent of 10 glasses. This move allowed her to represent the decomposition of numbers that underlies the reasonableness of the multiplication algorithm. At the same time, she was incorporating student ideas into the process of constructing and using representational tools in doing mathematics.[14]

The ways in which Smith, Rosen, and Lampert approach the teaching of multiplication also reflect and portray to students different views of what counts as "mathematics"--that is, what students are supposed to learn, what matters about learning mathematics, what it means to know and to do mathematics, and where the authority for truth lies. In both Smith's and Rosen's classes, learning multiplication means learning to calculate; mathematics is thereby synonymous with computation. Students are taught the computational algorithm which they practice so that they will memorize the procedure and increase their speed and accuracy in using it. Neither the meaning of the concepts nor the principles underlying the procedure were addressed. In this kind of teaching, knowing mathematics means remembering definitions, rules, and formulas and doing mathematics is portrayed as a straightforward step-by-step process. The goals derive from the school curriculum and what students need in order to move on. Epistemic authority rests with the teacher, who gives explanations and evaluates the correctness of students' answers.

In Lampert's classroom, students encounter a different view of mathematics. While she teaches the required fourth-grade curriculum, the ways in which she approaches it are colored by what she thinks is central to knowing mathematics. On one hand, she emphasizes meaningful understanding. Students are helped to acquire knowledge of concepts and procedures, the relationships among them, and why they work. Although she is teaching the same common fourth-grade topic as Smith and Rosen, her goals are different. Learning about multiplication is valued more for what students can learn about numbers, numeration, and operations with numbers than as an end in itself.

On the other hand, she also explicitly emphasizes the nature and epistemology of mathematics. Just as central as understanding mathematical concepts and procedures is understanding what it means to *do* mathematics, being able to validate one's own answers, having opportunities to engage in mathematical argument, and seeing value in mathematics beyond its utility in familiar everyday settings. Lampert (in press) discusses how the substantive and epistemological dimensions of mathematical knowledge go hand in hand in this view of mathematics. She explains that she tries to

> shift the locus of authority in the classroom--*away from* the teacher as a judge and the textbook as a standard for judgment and *toward* the teacher and students as inquirers who have the power to use mathematical tools to decide whether an answer or a procedure is reasonable.

But, she adds, students can do this only if they have meaningful control of the ideas:

> Students will not reason in mathematically appropriate ways about objects that have no meaning to them; in order for them to learn to reason about assertions involving such abstract symbols and operations as .000056 and $a^2 + b^3$, they need to connect these

symbols and operations to a domain in which they are competent to "make sense."

Teaching Mathematics: An Interaction Among Subject Matter Understandings, Views of Teaching, Learning, Learners, and Context

These kinds of teaching differ not only in what counts as knowledge of mathematics, but also in their assumptions about the teaching and learning of mathematics: about pupils, teachers, and the context of classrooms. What each of these teachers does is a function of the interactions among these understandings, assumptions, and beliefs. Smith's eye is on the fourth-grade curriculum; she feels responsible for her students' mastering the required material in order to go on to the next grade. For Smith, knowing math means *remembering* procedures and her teaching approach is based on the assumption that mathematics is learned through repeated practice and drill. She sees her role as showing pupils how to do the procedures, assigning and carefully monitoring their practice, and remediating individual students who have difficulty.

Like Smith, Rosen also believes that learning mathematics requires repeated practice. For both teachers, teaching multiplication begins with explanation and demonstration; the rest of the unit consists of practicing the procedure. Rosen, however, is more influenced by her view of her students than is Smith. Because Rosen believes her students to be weak, even "math disabled," she takes a more directive role throughout the practice phase than does Smith. This includes giving pupils tricks, mnemonics, and shortcuts, as well as walking them through the procedures over and over. Smith's classroom is very quiet; she assumes that pupils are engaged, and worries less than Rosen about keeping them in contact with the content. Rosen, who believes her students to be highly distractable and prone to behavior problems, plans activities which control her pupils' engagement with the subject matter. In both classes, pupils are expected to absorb and retain what they have been shown.

Lampert makes very different assumptions than either Smith or Rosen about what there is to be learned and why, as well as about how fourth graders can learn mathematics. She assumes that students must be actively involved in constructing their own understandings and meanings both individually and in groups. Practice takes on an entirely different meaning in this approach than in either of the other two approaches. Here, instead of a learning view of practice--that is, practicing mathematics by doing repeated examples of the skill being taught--students engage in a disciplinary view of practice: the practice *of* mathematics. Class activities are designed to involve the students in what it means to think about and do mathematics as mathematicians do (Collins, Brown, and Newmann, in press; Lave, 1987).

Lampert's view of her role appears to grow out of the interaction of her constructivist assumptions about learning and her disciplinary focus. With a goal of involving students in mathematical community, she must strive for a balance between helping students acquire established mathematical knowledge and encouraging them to invent and construct ideas themselves. Lampert believes, therefore, that the teacher has a critical role to play in facilitating students' mathematical learning. She introduces a variety of representational systems which can be used to reason about mathematics, models mathematical thinking and activity, and asks questions that push students to examine and articulate their ideas.

However, perhaps most significant in the classroom context is her role in guiding the direction, balance, and rhythm of classroom discourse by deciding which points the group should pursue, which questions to play down, which issues to table for the moment, decisions which she makes based on her knowledge of mathematics. The classroom group is critical in Lampert's approach for it represents the mathematical community within which students must establish their claims. In Smith's and Rosen's classes, learning mathematics is considered an individual matter; the group is a feature of the classroom context to be managed in fostering individual learning. Smith, in fact, prefers to "individualize" rather than to work as a group.

Subject Matter Knowledge: A Term in the Pedagogical Equation

A teacher's understanding of mathematics is a critical part of the resources available which comprise the realm of *pedagogical possibility* in teaching mathematics. A teacher cannot explain to her students the principles underlying the multiplication algorithm if she does not explicitly understand them herself. The representations she chooses may be mathematically misleading or may even fail to correspond at all. Yet a teacher who does understand the role of place value and the distributive property in multiplying large numbers will not necessarily draw upon this understanding in her teaching, if her ideas about learners or about learning intervene.

If she thinks, for example, that fourth graders will not profit from such knowledge, or that procedural competence should precede conceptual understanding in learning mathematics, she may choose to emphasize memorization of the algorithm. Two teachers who have similar understandings of place value and numeration may teach very differently based on differences in their assumptions about the teacher's role. One may talk directly about place value and explicitly show pupils what the digits in each place of a numeral represent. The other may engage students in a counting task which is designed to help them discover the power of grouping. These differences are a function of different assumptions about the teaching and learning of mathematics.

Still, a teacher who lacks Lampert's disciplinary knowledge of mathematics will not be able to teach as she does, for her approach to teaching is not possible without that kind of understanding of and about mathematics. Making the judgments about which student suggestions to pursue, developing the tasks that encourage certain kinds of exploration, and conducting fruitful class discussions--all these tasks depend heavily on the teacher's subject matter knowledge.

Are all these domains--subject matter, teaching and learning, learners, and context--coequally influential in teaching mathematics? Or does one domain tend to drive and shape a particular teacher's approach? Rosen, for example, seems to start from her ideas about her pupils. Her knowledge of mathematics, her view of her role, and her assumptions about learning all appear to be shaped by that starting point. Lampert's approach, however, seems clearly rooted in the subject matter; the pedagogy follows. In order to understand the role of subject matter knowledge in teaching mathematics, we need to explore the balance and interaction among the critical domains in different teachers' teaching of mathematics. This includes closely examining teachers' knowledge of and about mathematics as well as how that knowledge shapes or is shaped by their other ideas and assumptions.

The other side of the pedagogical equation is student learning. Studying the whole equation--from teacher knowledge to teacher thinking to teacher actions to student learning--is an agenda to which we must return. Research currently underway tends to focus on only part of the equation. This is appropriate, for in order to understand the subtle relationships among the terms, we need better ways of thinking about and studying each part of the equation. Past efforts often came up short as a result of unexamined or simplistic assumptions about subject matter knowledge, teaching, or student learning, or about the relationship among them. Still, we must keep our eye on the whole equation, for it is in studying these relationships that we will better understand what goes into teaching mathematics effectively.

We also need to pursue similar questions in research on teacher learning if teacher education is to be a more effective intervention in preparing people to teach mathematics well (Ball, 1988b). What do prospective teachers bring with them to teacher education? How do the ideas and understandings across these domains grow and change over time? We need to investigate the relative contributions of teachers' own schooling in mathematics, formal teacher education, and teaching experience to their subject matter knowledge and their approaches to teaching mathematics.

This paper ends with the reflection from a prospective teacher who, in trying to teach the concept of permutations, had abruptly discovered that he needed to revise his assumptions about learning to teach mathematics:

> When I decided to be a teacher, I knew there were a lot of things I had to learn about teaching, but I felt I knew everything there was to teach my students. During the permutation activities, I found I was as much a learner of subject matter as I was a learner of the art of teaching. My education in the future will not be limited to "how to teach," but also what it is I'm teaching. My knowledge of math must improve drastically if I am to teach effectively. (Ball, 1988b)

Like many people, he had taken subject matter knowledge for granted in teaching mathematics. To sum up, three points discussed in this chapter call this assumption sharply into question. First, learning to do mathematics in school, given the ways in which it is typically taught, may not equip even the successful student with adequate or appropriate knowledge of *or* about mathematics. Second, knowing mathematics for oneself may not be the same as knowing it in order to teach it. While tacit knowledge may serve one well personally, explicit understanding is necessary for teaching. Finally, subject matter knowledge does not exist separately in teaching, but shapes and is shaped by other kinds of knowledge and beliefs.

Footnotes

[1] Schwab (1961/1978) refers to this as knowledge of the *syntax* of the discipline.

[2]The results in this section are drawn from my dissertation research and all names used are pseudonyms. In this study, 19 prospective elementary and secondary teachers were interviewed at their point of entry into formal teacher education. I asked them questions to uncover what they knew and believed about mathematics, about teaching and learning math, about students, and about learning to teach. The goal of the research was to learn about the knowledge and beliefs of these 19 individuals as well as to contribute to the development of a theoretical framework for the question, "What do prospective teachers bring with them to teacher education that is likely to affect their learning to teach math?" (See Ball, 1988a).

[3]Following is a brief explanation of the substantive underpinnings of the question. It deals with four important ideas in mathematics: division, the concept of infinity, what it means for something to be "undefined," and the number 0. In addition, this question elicits respondents' ideas about mathematical knowledge: Is division by zero understood in terms of a rule--that is, "you can't divide by zero"--or is it logically related to the concept of division? Is the answer an arbitrary fact or a reasonable conclusion?

Division can be represented two ways:

1. I have seven slices of pizza. If I want to serve zero slices per person, how many portions do I have? (Answer: an infinite number of portions, or as many or as few portions as you like.)

2. I have seven slices of pizza. I want to split the pizza equally among zero (no) people. How much pizza will each person get? (Answer: This doesn't make sense. You actually aren't splitting, or dividing, the pizza at all.)

Note the two different meanings for division. In the first case, the referent for the answer is a *number of portions*; in the second case it is the *portion size*.

Since the second meaning for division does not make sense here, take a closer look at the first. What does it mean for there to be an infinite number of portions? In a way, it is a kind of oxymoron, for the idea of "portion" implies some way of dividing into a finite number of parts. Here the point is that you could have endless portions if a portion is "zero amount"--that is, you could go on "dividing" it forever and never finish.

The idea that one could "divide" seven forever conflicts with the *definition* of division--that is, that dividing something into some finite number of equal parts that, when recombined, form the whole. Dividing 10 into five groups, for instance, yields groups of two. One can reverse the process: five portions of two equals the original quantity 10 (5 x 2 = 10). Dividing seven by zero does not work this way, however. To divide seven by zero, theoretically, one could divide as long as one wishes. One might decide to stop at 15 or 710 or 5,983 groups of zero. Yet, there is no number of portions of zero that can be recombined to total seven--that is, there is no number that can be multiplied by zero to equal seven. Therefore, division by zero is actually *undefined*--it does not fit the *definition* of division.

[4]I borrow the term "algorithm rhyme" from Blake and Verhille, 1985.

[5]A provocative finding was the lack of difference by level between the responses of secondary teacher candidates, who are math majors, and of elementary teacher candidates, who are not. This issue and its implications are taken up in Ball (1988a).

[6]Researchers are currently pursuing critical questions about the relationship between conceptual and procedural knowledge in mathematics. See, for example, Hiebert and Lefevre, 1986.

[7]This requirement of explicit understanding holds even for teachers who do not choose to teach by telling. Facilitating students' construction of mathematical understanding, for instance, involves selecting fruitful tasks, asking good questions, and encouraging helpfully. In order to do this well, teachers must know what there is to be learned.

[8]The distinction between tacit and explicit ways of knowing is not intended as a dichotomy, but rather as a qualitative dimension along which understanding varies.

[9]The use of "conceptual" language creates significant problems of interpretation. In the multiplication question, discussed above, many of the teacher candidates mentioned "places" and yet did not necessarily seem to focus on place *value*. In the subtraction task, most of the prospective teachers *did* seem to be talking about place value--tens and ones. On one hand, this suggests that they did have some explicit understanding of the decimal numeration system.

On the other hand, unlike the steps of the multiplication algorithm, the steps of the "borrowing rhyme" refer explicitly to tens and ones (e.g., "borrow one from the tens, move it to the ones"). This may explain why teacher candidates seemed to focus more on place value when they talked about subtraction with regrouping.

Mention of "tens and ones" may be more procedural than conceptual, however. "Borrow one from the tens" is an ambiguous statement. In the example on page 25 it may mean, literally, take 1 away from the number in the tens *place*--that is, cross out the 6 and make it a 5. Or it may mean take 1 *ten* away from 6 tens, leaving 5 tens. Several responses suggested that referring to ones and tens is possible without engaging the concept of grouping (and regrouping) by tens, just as reference to *places* in the multiplication algorithm may not signal attention to place *value*.

[10]Critical to note here is that the standard school mathematics curriculum to which most prospective teachers have been subjected treats mathematics as discrete bits of knowledge. The results of these interviews with prospective teachers reflect in large measure the way in which mathematics is taught in this country.

[11]Bridget Smith is a pseudonym. The data about this teacher are part of the Teacher Education and Teacher Learning Study currently being conducted by the National Center for Research on Teacher Education. For more information about the study and, in particular, about the theoretical framework and instrumentation of the research, see Ball and McDiarmid (in press) or NCRTE (1988).

[12]Belinda Rosen is a pseudonym. These data are also part of the Teacher Education and Teacher Learning Study being conducted by the NCRTE.

[13]Magdalene Lampert (her real name) teaches fourth- and fifth- grade mathematics and is also a university professor and researcher. The material in this section is drawn from her own writing about her teaching (Lampert, 1986, in press).

[14]This provides a striking contrast with the way in which many prospective teachers choose representations. They tend to focus more on using media which will appeal to students (e.g., candies) and often neglect to consider the mathematical appropriateness of the representation or its helpfulness in teaching.

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1.Schwab (1961/1978) refers to this as knowledge of the *syntax* of the discipline.

2.The results in this section are drawn from my dissertation research and all names used are pseudonyms. In this study, 19 prospective elementary and secondary teachers were interviewed at their point of entry into formal teacher education. I asked them questions to uncover what they knew and believed about mathematics, about teaching and learning math, about students, and about learning to teach. The goal of the research was to learn about the knowledge and beliefs of these 19 individuals as well as to contribute to the development of a theoretical framework for the question, "What do prospective teachers bring with them to teacher education that is likely to affect their learning to teach math?" (See Ball, 1988a).

3.Following is a brief explanation of the substantive underpinnings of the question. It deals with four important ideas in mathematics: division, the concept of infinity, what it means for something to be "undefined," and the number 0. In addition, this question elicits respondents' ideas about mathematical knowledge: Is division by zero understood in terms of a rule--that is, "you can't divide by zero"--or is it logically related to the concept of division? Is the answer an arbitrary fact or a reasonable conclusion?

Division can be represented two ways:

> 1. I have seven slices of pizza. If I want to serve zero slices per person, how many portions do I have? (Answer: an infinite number of portions, or as many or as few portions as you like.)
>
> 2. I have seven slices of pizza. I want to split the pizza equally among zero (no) people. How much pizza will each person get? (Answer: This doesn't make sense. You actually aren't splitting, or dividing, the pizza at all.)

Note the two different meanings for division. In the first case, the referent for the answer is a *number of portions*; in the second case it is the *portion size*.

Since the second meaning for division does not make sense here, take a closer look at the first. What does it mean for there to be an infinite number of portions? In a way, it is a kind of oxymoron, for the idea of "portion" implies some way of dividing into a finite number of parts. Here the point is that you could have endless portions if a portion is "zero amount"--that is, you could go on "dividing" it forever and never finish.

The idea that one could "divide" seven forever conflicts with the *definition* of division--that is, that dividing something into some finite number of equal parts that, when recombined, form the whole. Dividing 10 into five groups, for instance, yields groups of two. One can reverse the process: five portions of two equals the original quantity 10 (5 x 2 = 10). Dividing seven by zero does not work this way, however. To divide seven by zero, theoretically, one could divide as long as one wishes. One might decide to stop at 15 or 710 or 5,983 groups of zero. Yet, there is no number of portions of zero that can be recombined to total seven--that is, there is no number that can be multiplied by zero to equal seven. Therefore, division by zero is actually *undefined*--it does not fit the *definition* of division.

4.I borrow the term "algorithm rhyme" from Blake and Verhille, 1985.

5.A provocative finding was the lack of difference by level between the responses of secondary teacher candidates, who are math majors, and of elementary teacher candidates, who are not. This issue and its implications are taken up in Ball (1988a).

6.Researchers are currently pursuing critical questions about the relationship between conceptual and procedural knowledge in mathematics. See, for example, Hiebert and Lefevre, 1986.

7.This requirement of explicit understanding holds even for teachers who do not choose to teach by telling. Facilitating students' construction of mathematical understanding, for instance, involves selecting fruitful tasks, asking good questions, and encouraging helpfully. In order to do this well, teachers must know what there is to be learned.

8.The distinction between tacit and explicit ways of knowing is not intended as a dichotomy, but rather as a qualitative dimension along which understanding varies.

9.The use of "conceptual" language creates significant problems of interpretation. In the multiplication question, discussed
above, many of the teacher candidates mentioned "places" and yet did not necessarily seem to focus on place *value*. In the
subtraction task, most of the prospective teachers *did* seem to be talking about place value--tens and ones. On one hand, this
suggests that they did have some explicit understanding of the decimal numeration system.

On the other hand, unlike the steps of the multiplication algorithm, the steps of the "borrowing rhyme" refer explicitly
to tens and ones (e.g., "borrow one from the tens, move it to the ones"). This may explain why teacher candidates seemed
to focus more on place value when they talked about subtraction with regrouping.

Mention of "tens and ones" may be more procedural than conceptual, however. "Borrow one from the tens" is an
ambiguous statement. In the example on page 25 it may mean, literally, take 1 away from the number in the tens *place*-
that is, cross out the 6 and make it a 5. Or it may mean take 1 *ten* away from 6 tens, leaving 5 tens. Several responses
suggested that referring to ones and tens is possible without engaging the concept of grouping (and regrouping) by tens, just
as reference to *places* in the multiplication algorithm may not signal attention to place *value*.

10.Critical to note here is that the standard school mathematics curriculum to which most prospective teachers have been subjected treats mathematics as discrete bits of knowledge. The results of these interviews with prospective teachers reflect in large measure the way in which mathematics is taught in this country.

11.Bridget Smith is a pseudonym. The data about this teacher are part of the Teacher Education and Teacher Learning Study currently being conducted by the National Center for Research on Teacher Education. For more information about the study and, in particular, about the theoretical framework and instrumentation of the research, see Ball and McDiarmid (in press) or NCRTE (1988).

12.Belinda Rosen is a pseudonym. These data are also part of the Teacher Education and Teacher Learning Study being conducted by the NCRTE.

13.Magdalene Lampert (her real name) teaches fourth- and fifth- grade mathematics and is also a university professor and researcher. The material in this section is drawn from her own writing about her teaching (Lampert, 1986, in press).

14.This provides a striking contrast with the way in which many prospective teachers choose representations. They tend to focus more on using media which will appeal to students (e.g., candies) and often neglect to consider the mathematical appropriateness of the representation or its helpfulness in teaching.