THE SUBJECT MATTER PREPARATION OF PROSPECTIVE MATHEMATICS TEACHERS: CHALLENGING THE MYTHS

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> I am really worried about teaching something to kids I may not know. Like long division--I can *do* it--but I don't know if I could really *teach* it because I don't know if I really *know* it or know how to *word* it. (Cathy, elementary teacher candidate)

> Teaching the material is no problem. I have had *so* much math now--I feel very relaxed about algebra and geometry. (Mark, prospective secondary mathematics teacher)

> I'm not scared that kids will ask me . . . a *computational* question that I cannot solve, I'm more worried about answering *conceptual* questions. Right now, my biggest fear--and I'm going to have to confront this on the 3rd of February--is what I am going to do if they ask me some kind of question like, "Why are there negative numbers?" (Cindy, prospective secondary mathematics teacher)[2]

Cathy, Mark, and Cindy, all preservice teachers, differ in what they think they need to know in order to teach mathematics. While Mark has confidence in the sufficiency of his mathematics knowledge, both Cindy and Cathy suspect that they may come up short when they try to teach. These three teacher candidates represent alternative points of view about the subject matter preparation of teachers. Cathy's view--that she understands the mathematics herself, but needs to learn to teach it--is the basis for traditional formal preservice teacher education. Mark expresses a view that undergirds many of the current proposals to reform teacher education: that people who have majored in mathematics are steeped in the subject matter and have thus acquired the subject matter knowledge needed to teach. Cindy's fear that, although she can do the mathematics, she may not have the kind of mathematical understandings she will need in order to help students learn, is insufficiently shared by those who consider the preparation and certification of teachers.

The mathematics knowledge of prospective teachers is the focus of this paper. Despite the fact that subject matter knowledge is logically central to teaching (Buchmann, 1984), it is rarely the object of adequate consideration in preparing or certifying teachers. Three widely held assumptions help to explain this odd state of affairs. First, policymakers and teacher educators seem to assume that topics

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[2]All names used are pseudonyms.

such as place value and division, fractions and ratio, and measurement and equations are "basic" and commonly understood. Implicitly, the message is: If you can "do" them correctly--if you can get the right answers--then you can teach these topics. This assumption holds that *remembering* and *doing* are the critical correlates of mathematical understanding.

The first assumption leads logically to the second. If the content of school mathematics is simple and commonly understood, then prospective teachers do not need to relearn the stuff of the elementary and secondary curriculum. Prospective teachers study little school mathematics content as part of their formal preparation for teaching, a fact that indicates the prevalence of this second assumption. Although many mathematics educators do include mathematical content in their methods classes, they often concentrate on nontraditional content, such as probability or permutations. While they recognize that their students will teach multiplication as well as probability, they choose to emphasize novel content instead of revisiting the old, presumably familiar, content. Whatever the contributions of upper level mathematics study to teachers' disciplinary knowledge, the fact remains that a large part of what they teach is material which they last studied in elementary and secondary school.

The third assumption has to do with the outcomes of upper level mathematics study. Many recent proposals for reforming teacher education and certification (e.g., Carnegie Task Force on Teaching as a Profession, 1986; Holmes Group, 1986) recommend that elementary teachers specialize in an academic discipline. Other current reforms propose to certify college graduates who have completed an academic major but have had no teacher education. Underlying such proposals is the assumption that the study entailed in taking college-level mathematics will equip the prospective teacher with a deep and broad understanding of the subject matter.

This paper challenges these prevalent assumptions about the subject matter preparation of mathematics teachers by examining the knowledge of a sample of prospective elementary and secondary teacher education students. An examination of the mathematical understandings these students brought with them to teacher education raises serious questions about their subject matter preparation for mathematics teaching. The data highlight the need to reexamine common assumptions about what prospective teachers need to know and how they can learn that, assumptions that underlie current teacher education practice as well as, paradoxically, proposals to reform teacher preparation.

Underlying the argument in this paper is an assumption that the goal of mathematics teaching is to develop mathematical understanding. On one hand, this implies that pupils should acquire knowledge of mathematical concepts and procedures, the relationships among them, and why they work. On the other hand, understanding equally implies learning about mathematical ways of knowing as well as about mathematical substance. The two are intertwined: In order for students to develop power and control in mathematics, students must learn to validate their own answers. They must have opportunities to make conjectures, justify their claims, and engage in mathematical argument, all of which both depend upon and can extend pupils' understandings of concepts and procedures (Lampert, 1986, 1988).

Method

This paper draws on interviews conducted with prospective elementary and secondary teachers at Michigan State University. Nineteen teacher education students--10 elementary and 9 secondary--were interviewed at the point at which they were about to enter their first education course. The elementary teacher education students were majoring in elementary education or child development and had no disciplinary specialization. The secondary teacher education students were mathematics majors or minors. All participants were selected from among a group of volunteers recruited from sections of the introductory teacher education course.

The goal of the research was to develop a theoretical framework for assessing what teacher candidates bring to their formal preparation to teach mathematics by examining the knowledge and beliefs of these 19 students. Using the process of "modified analytic induction" (Bogdan and Biklen, 1982), the intent was to use the data collected to revise and reformulate a preliminary framework for examining the knowledge and reasoning of beginning mathematics teachers.

Since the purpose of conducting the interviews was to learn the range and diversity of what prospective teachers bring to teacher education that might affect their learning to teach mathematics, the sample was selected to vary on several key criteria: gender, academic history in college mathematics, and stated feelings about mathematics (based on an item posed on the volunteer form).[3] A two-part interview was developed to learn what the prospective teachers knew and believed, as well as how they thought and felt about mathematics, about the teaching and learning of mathematics, and about students as learners of math. The first interview explored the prospective teacher's personal history and his or her ideas about mathematics, teaching and learning math, and self. The tasks and questions in the second session were grounded in scenarios of classroom teaching and woven with particular mathematical topics.[4]

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[3]There are other variables potentially connected to what prospective teachers bring with them to mathematics teacher education--high school mathematics experience, age, or whether or not they attended a community college prior to coming to the university, for instance. The sample used in this study was not large enough to justify stratifying it along many dimensions. Additional academic data was collected about the teacher candidates which, in analyzing interview responses, may suggest possible connections worth systematic exploration in the future. Bogdan and Biklen (1982) explain the rationale for the *purposeful sampling*:

> This research procedure insures that a variety of types of subjects are included, but it does not tell you how many, nor in what proportion the types appear in the population. . . . You choose particular subjects to include because they are believed to facilitate the expansion of the developing theory. (pg. 67)

[4]Among these were rectangles and squares, perimeter and area, place value, subtraction with regrouping, multiplication, division, fractions, zero and infinity, proportion, variables and solving equations, theory and proof, slope and graphing.

The interviews were tape-recorded and transcribed. Drawing from careful substantive analyses of each interview question, a set of response categories was created for each one. These categories were modified in the course of analyzing the data to accommodate better the kinds of responses people gave. Summary analytic tables for each question were constructed. Most questions were cross-analyzed on several dimensions: subject matter understanding, ideas about teaching, learning and the teacher's role, and feelings or attitudes about mathematics, pupils, or self.

Three purposes underlie the examination of prospective teachers' substantive knowledge of division--one of the focal topics of the study: (a) to provide a portrait of the understanding of division held by these 19 students; (b) to illustrate an approach to the examination and analysis of teachers' substantive knowledge of mathematics; and (c) to challenge common assumptions about subject matter knowledge and the preparation of mathematics teachers.

Examining "Subject Matter Knowledge"

Subject matter knowledge, although attracting increased attention (Shulman, 1986), is presently mired in a morass of differing conceptions and definitions (Ball, in press-b). This paper deals directly with one aspect of prospective teachers' subject matter knowledge--substantive knowledge--and, more subtly, with a second--their knowledge *about* mathematics. Substantive knowledge refers to understandings of particular topics, procedures, and concepts, and the relationships among these (Davis, 1986; Hiebert and Lefevre, 1986; Skemp, 1978). Knowledge about mathematics includes understandings about the nature of knowledge in the discipline--where it comes from, how it changes, and how truth is established; the relative centrality of different ideas, as well as what is conventional or socially agreed upon in mathematics versus what is necessary or logical.[5] The results discussed below show that people's assumptions about the nature of mathematical knowledge shape their understandings of and approach to the subject. The prospective teachers' substantive knowledge was analyzed along three qualitative dimensions: (a) truth value; (b) legitimacy; and (c) connectedness.

[5]Schwab (1961/1978) refers to these kinds of understandings as knowledge of the substantive and syntactic structures of a discipline--in this case, mathematics.

Truth Value

In the work reported here, at different times, the terms "knowledge," "understanding," "belief," and "idea" are used to refer to what the prospective teachers "knew." Some prospective teachers, for instance, think that $7 \div 0 = 0$, that squares are not rectangles, and that doing mathematics means adding, subtracting, multiplying, and dividing (Ball, 1988). They believe these "facts" to be true and treat them as knowledge. At the same time, the truth value of prospective teachers' ideas must be an important criterion for appraising their subject matter understandings for, as teachers, they will be responsible for helping their pupils acquire disciplinary knowledge. Truth value, means, in part, mathematical correctness.

Correctness, however, is a slippery idea in mathematics. Resnick and Ford (1981) specify correctness to be that which is "defined by the consensus of mathematicians" (p. 206). Although it seems reasonable, this definition ignores two critical issues. First, mathematicians do not reach consensus on some questions. And second, ideas are not absolutely true in mathematics (Davis and Hersh, 1981; Kline, 1980). Often they depend on the particular context. Furthermore, even some ideas about which many mathematicians agree are ultimately fallible (Lakatos, 1976). Having "correct" knowledge, therefore, entails knowing the conditions and limits of an idea. Parallel lines never meet--but this "fact" is not true in non-Euclidean geometry. First-grade teachers may tell pupils that 0 is the smallest number or that 3 is the next number after 2--but these ideas are true only if the domain is the counting numbers.

Thus, mathematical correctness depends on the qualifications placed on its truth value.[6] In teaching this is additionally important, for pupils' developing understandings often follow the historical evolution of mathematics. Ninth graders operating in the domain of rational numbers, for example, are likely to believe that there is no "smallest number" and "next number" after 2. Does this make the first graders wrong if they believe that 3 comes after 2? And what if a pupil makes an assertion that presses on the boundaries of the current domain? Suppose a first grader claims that 2½ is the next number after 2? Epistemological dilemmas such as this one arise in everyday teaching; figuring out how to deal with them is central to teaching mathematics for understanding. In order to be able to decide how to respond, teachers' own understandings of mathematics must be appropriately qualified. Therefore, for the first criterion in assessing the correctness of teacher candidates' knowledge, I also appraised the degree to which, where needed, they qualified their ideas.

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[6]The term qualification has been borrowed and adapted from Wilson (1988). Writing about subject matter knowledge in history, she refers to the ways in which historians limit, or qualify, their claims as "qualification." The particular aspects of qualification that she focuses on--contextualization and the underdetermined nature of historical knowledge--are specific to history. They offer an interesting contrast with the aspects central to mathematical knowing: the relationship of validity to context and fallibility (see Wilson and Ball, in preparation).

Legitimacy: Justification and Explanation

Knowledge is also not an all-or-nothing matter (Nickerson, 1985). What does one say about the knowledge of a person who says that "you can't divide by zero"? This is true, but of interest here is also how he understands it--as an arbitrary "fact" or as a logical consequence of other mathematical ideas and principles. He may, by way of explanation, say that "it's just one of those things you have to remember," "zero can't do anything to a number," or "it's undefined." Or he may prove his assertion by comparing division by 0 to division by 2 or by using the inverse relationship of multiplication and division. Each case would reveal significantly different things about his understanding of division by zero as well as his notions about mathematics.

Using Scheffler's (1965) definition of knowledge as "justified true belief," the role of justification as a dimension of understanding is discussed next. To get below the level of right answers, uncovering what people know in mathematics is an endeavor fraught with practical and conceptual difficulties. Not the least of these is the problem of inferring what people understand from what they do or say. The assumption that people understand the underlying principles of procedures that they have learned to perform is questionable (Hatano, 1982). One math major reflected when he tried to explain the basis for the algorithm for multiplying multidigit numbers, "I absolutely do it by the rote process--I would have to think about it". Not clear, however, was whether he had ever understood the underlying principles of multiplication--and had simply forgotten them--or whether these were never known, never considered--and would therefore have to be figured out or learned. Certainly many children and adults go through mathematical motions without ever understanding the underlying principles or meaning. How many people, for example, can say why "cross-multiply and divide" works? They simply do it; their knowledge is, however, algorithmic and unwarranted; that is, they believe that it is true or right, but are unable to justify it mathematically. Of course, some mathematical understanding may be tacit. Successful mathematicians can unravel perplexing problems without being able to articulate all of what they know. Not unrelated to Schon's (1983) "knowing-in-action," the mathematicians' work reflects both tacit understanding and intuitive and habituated actions (Noddings, 1985). Experts in many domains, while able to perform skillfully, may not always be able to specify the components of or bases for their actions. Their activity nevertheless implies knowledge.

The math major unable to explain why the numbers move over in the partial products in the multiplication algorithm may know, at some level, that multiplying by the "2" in

$$\begin{array}{r} 51 \\ \underline{\times\ 24} \end{array}$$

is to multiply by *20*. If all he does, however, is produce the correct answer--1,224--and if all his explanation consists of is to say is that "you have to line it up under the 2 because that's what you're multiplying by," then the extent to which he does or does not understand the meaning is unclear.

Analyzing people's knowledge is of course complicated by the extent to which they are able to articulate or otherwise access that knowledge. Tacit or buried knowledge, whatever its role in independent mathematical activity, however, is inadequate for teaching. In order to help someone else understand and do mathematics, being able to do it oneself is not sufficient. A necessary level of knowledge for teaching involves being able to explain mathematics and to be able to access those explanations when needed.

Explanations of mathematics entail more than repeating the words of mathematical procedures or definitions. The statement, for example, that you "carry the 1" is not a mathematical explanation of regrouping in addition; neither, by itself is the statement "7 ÷ 0 is undefined." To explain mathematics is to focus on the meaning, on the ideas and concepts. To explain is to say why, to justify the logic, or to identify the convention. Being able to explain mathematics is essential knowledge even for teachers who do not teach mathematics in a show-and-tell mode, for explanation requires explicit understanding, not just remembering or doing. In order to facilitate students' construction of mathematical understanding, teachers must select fruitful tasks, ask good questions, and judge which student ideas are most worth pursuing. All of these demand principled and warranted knowledge of mathematics: explicit recognition and understanding of underlying ideas. Mathematical explanation, necessarily grounded in meaning, represents the kind of legitimate knowledge needed to teach students to understand. Consequently, legitimacy is used as a second criterion for appraising the quality of prospective teachers' subject matter understandings.

Connectedness

A third qualitative dimension of mathematical knowledge relates to the connections among ideas. However well explained or correct, mathematical knowledge is not a collection of disparate facts and procedures. Connections exist at multiple levels between and among ideas. Smaller ideas belong to various families of larger concepts; for example, decimals are related to fractions as well as to base ten numeration and place value. Topics are connected to others of equivalent size; addition, for instance, is fundamentally connected to multiplication. Elementary mathematics links to more abstract content--algebra is a first cousin of arithmetic, and the measurement of irregular shapes is akin to integration in calculus. Mathematical ideas can be linked in numerous ways; no one right structure or map exists.

In the school mathematics curriculum, however, mathematics is delivered in compartments separated in time and meaning. Even content taught within the same grade or course is often fragmented; rarely are students encouraged to make connections among the ideas they encounter in school. The standard school mathematics curriculum treats mathematics as a collection of discrete bits of procedural knowledge. So extreme is this fragmentation traditionally that, for instance, two-digit addition is taught at a different time than is three-digit addition; addition prior to subtraction.

This tendency to compartmentalize mathematical knowledge substantially increases the cognitive load entailed in knowing and using mathematics. Each idea or procedure seems to be a separate case. Each requires a different rule, all of which must be individually memorized and accessed. "Knowing" mathematics is easily reduced to the senseless activity of a "wild goose chase" after right answers (Erlwanger, 1975). The connections that students make affect the integration and accessibility of their knowledge (Greeno, 1978).

In addition to making it harder to learn, treating mathematics as a collection of separate facts and procedures also seriously misrepresents the logic and nature of the discipline to students. Convention is blurred with deduction; invention and argument do not figure in classroom discourse or in the outcomes of instruction (Ball, 1988; Lampert, 1988). If teachers are to break away from this common approach to teaching and learning mathematics and teach for understanding instead, they must have a connected rather than a compartmentalized knowledge of mathematics themselves. They must have a sense of the dynamic of mathematical knowing as well as of the static of the body of accumulated knowledge (Romberg, 1983). Therefore, the third criterion used in analyzing prospective teachers' knowledge was the degree to which they seem to make explicit connections among ideas.

Knowledge of Division

Why focus on division? Mathematics educators despair at the fact that children spend the better part of fourth and fifth grade "in" division--being trained to do what a $5 machine can do faster and more accurately (Schwartz, 1987). They urge teachers to emphasize division less, especially long division, and to teach better mathematical content instead.

Still, the concept of division is a central one in mathematics at all levels and it figures prominently throughout the K-12 curriculum. Furthermore, division is worthwhile content for what students can learn about rational and irrational numbers, about place value, about the connections among the four basic operations, as well as about the limits and power of relating mathematics to the real world. For these reasons, as well as the fact that students also often have difficulty learning it, division is a topic about which teachers should have true, legitimate, and connected knowledge.

In order to examine the connectedness of teacher candidates' knowledge of division, three different mathematical contexts were chosen: division with fractions, division by zero, and division with algebraic equations. These contexts, because they are separated in time and meaning by the school curriculum, do not appear obviously connected to teacher candidates. Yet division is the conceptual key to each. The kinds of tasks posed also invited the teacher candidates to display explicit conceptual understanding: They were asked to explain and to generate representations.

In order to help the reader appraise the prospective teachers' knowledge a brief discussion of division and its meaning in each of the three contexts is presented. This discussion also illustrates the three dimensions of substantive knowledge.

At its foundation, division has to do with forming groups. Two kinds of groupings are possible:

> 1. Forming groups of a certain *size* (e.g., taking a class of 28 students and forming groups of 4). The problem is how many groups of that size can be formed? This is sometimes referred to as the measurement model of division.
>
> 2. Forming a certain *number* of groups (e.g., taking a class of 28 students and forming 4 groups). The problem is to determine the size of each group. This model is sometimes referred to as the partitive model of division.

Consider a typical division statement with whole numbers, such as 7 ÷ 2. What does this mean? It may represent one of two kinds of situations:

> 1. I have 7 slices of pizza. If I want to serve 2 slices per person, how many portions do I have? (Measurement interpretation--Answer: 3½ portions)
>
> 2. I have 7 slices of pizza. I want to split the pizza equally between 2 people. How much pizza will each person get? (Partitive interpretation--Answer: 3½ slices)

Notice that because these two situations represent two different meanings for division, the referent for 3½ is different in the two cases. In the first situation, the answer is a number of *portions* (the size of the portion was already decided); in the second, the answer is a number of slices *per portion* (the question was how large each portion would be). In each case, multiplying the result (3½) by the number used to divide the original total yields that total (3½ x 2 = 7).

Remembering to "invert and multiply"--that is, to invert the divisor and multiply it by the dividend--is the traditional way of knowing division with fractions. A typical sixth-grade textbook page introducing division of fractions says simply, "Dividing by a fraction is the same as multiplying by its reciprocal." Little attention is given to the meaning of division with fractions and no connections are made between division with fractions and division with whole numbers. Each is treated as a special case.

Dividing by fractions is not different conceptually from dividing by whole numbers, however. Suppose, for example, that you owe a friend $100 and must repay the money without interest. You can explore how long it would take to repay this debt, given different payment amounts. If you pay $2 per week, it will take you 50 weeks. This can be formulated mathematically as 100 ÷ 2 = 50 (and 50 weeks x $2 = $100). Now, consider how long it will take you if you repay at a rate of 50 cents per week: 50 cents = ½ dollar, so this option can be expressed as 100 ÷ ½. It will take 200 weeks to repay the debt (200 x .50 = $100).[7] Since division with fractions is most often taught algorithmically, it is a strategic site for examining the extent to which prospective teachers understand the meaning of division itself (Davis, 1983).

Division by zero, the second case of division explored in this study, is also no more complicated than division with other rational numbers. The result, however, is different. Extending the debt example, suppose you owe $100 and want to know how long it will take you to repay your debt if you make payments of $0 per week. What is 100 ÷ 0? Obviously you will never repay your debt at that rate--you will be indebted forever. There is no solution to your indebtedness if you choose to repay in installments of $0. Going one step further, there is no number of weeks that you can multiply by $0 and come up with a total of $100. Yet, in the earlier examples, the answer could always be multiplied by the divisor and equal the original total. Division by zero is undefined: It has no solution that fits sensibly within the meaning of division and its relationship to multiplication.

The third context in which teacher candidates' knowledge of division was looked at was in algebraic equations. The equation **x/0.2**= 5 gives information that permits one to identify the correct value for an unknown number, denoted as x. In common language, the equation says that when one divides this unknown number by .2, one gets 5 as the answer. Knowledge of division makes clear what this means: that there are five groups of two-tenths in the number, or that the number is 2/10 of 5. Reasoning conceptually about division in this way allows one to identify the number without performing any manipulations on the equation. The answer is 1 since 1 can be divided into five groups of 0.2; 0.2 "goes into" 1 5 times.

Knowledge of Division in Division By Fractions

To learn about teacher candidates' knowledge of division, they were asked to develop a representation--a story, a model, a picture, a real-world situation--of the division statement 1 3/4 ÷ ½. The traditional algorithm for dividing fractions that most students learn in school is "invert and multiply"--that is, invert the divisor and multiply it by the dividend. In addition, any mixed numbers must be converted to improper fractions. 1 3/4 ÷ ½ becomes 7/4 x 2/1. Multiplying the numerators and denominators produces 14/4, which should be expressed as 3½.

An appropriate representation should show that the question is "how many ½'s are there in 1 3/4?" For example:

> A recipe calls for ½ cup butter. How many batches can one make if one has 1 3/4 cups butter? Answer: 3½ batches.

[7]It is worth noting that the procedure "invert and multiply" is not unique to dividing with fractions. For instance, 6 ÷ 2 yields the same result as 6 x ½ (where one has inverted the 2 and multiplied the result by the dividend). Yet rarely, if ever, is this made explicit to pupils.

Why? Because there are 3½ ½-cup portions of butter in 1 3/4 cups of butter. This story makes clear the referent for the answer 3½--it refers to 3½ *halves*.

The prospective teachers--the elementary candidates as well as the secondary students who were majoring in mathematics--had significant difficulty with the meaning of division with fractions. Few elementary or secondary teacher candidates were able to generate a mathematically appropriate representation of the division. These results fit with evidence from other parts of the interview that suggested that their substantive understanding of mathematics tended to be both rule-bound and compartmentalized. The teacher candidates' responses were categorized as appropriate, inappropriate, unable to generate a representation (See Table 1 for the distribution of responses by elementary and secondary teacher candidates).

Table 1

Division by Fractions: 1 3/4 ÷ 1/2

		Teacher Candidates (N=18)		
		Elementary	Secondary	TOTALS
Category of Response	Appropriate representation	0	5	5
	Inappropriate representation	3	2	5
	Unable to generate a representation	6	2	8
	TOTALS	9	9	

Appropriate representations. Five secondary teacher candidates were able to generate a completely appropriate representation of 1 3/4 ÷ ½; However, this did not come easily to any of them. For example, Terrell first said he couldn't "think of anything specific." Then he said he would use pizza:

> If you took the pizza and took one half of a pizza and you took a whole pizza and three quarters of a pizza [that would be one and three quarters]. You put the half of the pizza on top of each piece. So first you'd take the whole pizza and you'd put it on top of it. Then you'd take that off, whatever it fits on and you'd do it again. Only take it off if it fits the whole thing, if . . . both pieces are equal. Then you go through the half a piece and do the same for that. Take that off. Then you get that last piece and you . . . well, that's the way I'd explain it.

Terrell then explained what the answer (3½) meant in this context:

> You'd take the half and the answer would be how many times you got a whole half (if you want to say that). Of the . . . whatever's left over, what part of it is *of* the half, I guess you could say. You'd have a quarter left, which is half of a half.

While sometimes rather confusing to follow, these prospective teachers' responses did make mathematical sense. Their answers indicated that they did think in terms of how many halves there are in one and three-fourths. These students were in the minority, however.

Inappropriate representations. Five teacher candidates generated representations that did not correspond to the problem. The most frequent error was to represent division by 2 instead of division by ½. For example, Barb, a mathematics major, gave the following story:

> If we had one and three-quarters pizzas left and there were two of us dying to split it, then how would we be able to split that?

Answering her own questions, she said each person would get 7/8 of a pizza altogether. This error--representing division by 2 rather than division by ½--was the most common error among the students.

However, Allen, an elementary major with 27 credits in college mathematics (through calculus) had a different problem than that of the other teacher candidates:

> Somebody has one and three quarter apples or something like that and they wanted to . . . double the amount of apples they have . . . just give 'em an equation . . . using only fractions. Other than that I couldn't think of any situation where you could . . . logically divide a number like that by one half instead of just multiplying by two. You would have to . . . be working on dividing like fractions and setting up equations using that. Right, you just have to say. . . . "Well, you know, use this and

figure the story problem out but, only use fractions."

Allen's story modeled 1 3/4 x 2--the procedure used to divide fractions. Using the frame of reference of the procedure "invert and multiply," Allen did not seem to focus on the concept of division by ½.

The teacher candidates' comments showed that they saw the question as one about fractions instead of about division. When asked, for example, what made this difficult, most commented that it was hard (or impossible) to relate 1 3/4 ÷ ½ to real life because, as one said, "You don't think in fractions, you think more in whole numbers." Not only did their explanations reveal that they framed the problem in terms of fractions, but also that many were uncomfortable with fractions as real quantities. Several commented that they didn't "like" fractions. The prospective teachers also tended to confuse dividing *in* half with dividing *by* one-half and they did not seem to notice the difference even though the answer they got to their story problem (7/8) differed from their calculated answer (3½).

The teacher candidates did not notice this discrepancy because it was masked by a slippery change in the referent unit from wholes to fourths. Here is a typical example of the way they reasoned: Suppose you have 1 3/4 pizzas which you want to split equally between two hungry teenagers (1 3/4 ÷ 2). Each pizza is divided into 4 pieces, so you have 7 pieces. Therefore each person gets 7/8 of a pizza, which is *3½* pieces of pizza. However, to divide something *in* half means to divide it into two equal parts (÷ 2); to divide something *by* one-half means to form groups of ½.

The teacher candidates' error may have resulted from a common but troublesome confounding of everyday language with mathematical language. Orr (1987) writes about the mismatches between linguistic and mathematical use of prepositions. In this case, "6 divided into 2" can mean two things: six divided into two parts--($2\overline{)6}$)--or six into two--($6\overline{)2}$). Awareness of such confusions is essential for teachers if they are to help their pupils understand mathematics.

Stumped. Eight teacher candidates could not generate any representation, correct or incorrect, for 1 3/4 ÷ ½. Marsha, for example, said she felt stuck and couldn't even remember how to get the answer (i.e., through computation). She explained that she hadn't done this since high school:

> I don't know what I'm remembering here that I did, I found the common denominator and I did this, but I think what I have to do is go 4 and 1, 4 and then plus 3 is 7, fourths, no I think that's what I did, one-half, but then, see, I don't know what I need to divide. I don't even remember that. . . . I remember doing these for a long time though and trying to get these down, and so I remember bits and pieces and then I try to apply it generally, and I can't do it.

The prospective teachers who did not generate a representation at all seemed to fall into two groups. In one group, some did recognize the conceptual problem. They initially proposed stories or models which represented division by 2 and then realized this themselves. Others, however, seemed to think that it was not a feasible task, that 1 3/4 ÷ ½ could not *be* represented in real world terms. On one hand, those who recognized the conceptual issue (that this was about division by ½, which is not the same as division by 2) revealed a better grasp of the idea than those who constructed a story that represented division by 2. Still, despite this recognition, they were unable to figure out what division by ½ meant. On the other hand, those who thought it was an impossible task revealed a view of mathematics as a senseless activity, out of which meaning cannot necessarily be made.

The teacher candidates' understanding of division in division by fractions. Although few of the prospective teachers even mentioned division explicitly while talking about the fractions exercise, the difficulties all of them experienced (including those who succeeded in generating an appropriate representation) suggest a narrow understanding of division. While they worried about the fractions in the problem, they also only considered division in partitive terms: forming a certain number of equal parts. This model of division corresponds less easily to division with fractions than does the measurement interpretation of division.

In a study of preservice elementary teachers' understanding of division, Graeber, Tirosh, and Glover (1986) found that teacher candidates tended to think only in terms of this partitive interpretation. Few of the preservice teachers in their study were able to write story problems that modeled a measurement interpretation of division. This finding offers another insight into why the task of making meaning out of 1 3/4 ÷ ½ was so difficult for the prospective teachers in this study.

Knowledge of Division in Division by Zero

The teacher candidates' understanding of division was explored with a question about division by zero. While some might argue that this question deals with an esoteric little bit of mathematics, I contend that it deals with four important ideas in mathematics: division, the concept of infinity, what it means for something to be "undefined," and the number zero, all significant mathematical content. Prospective teachers' responses depended on their understanding of the specific content at hand as well as of mathematical knowledge and mathematical ways of knowing. When they are explaining, their responses reveal the legitimacy of their knowledge. What they focus on also provides information about what they think counts as an "explanation" in mathematics.

Truth and legitimacy. Of the 19 teacher candidates, 5 explained the meaning of division by zero. Most of the prospective teachers responded by stating a rule, 5 of which were incorrect. Two did not know. (See Table 2 for the distribution of types of responses.)

Table 2

Division by Zero: 7 ÷ 0

Category of Response		*Teacher Candidates (N=19)* Elementary	Secondary	TOTALS
Meaning		1	4	5
Rule	correct	2	5	12 (7 correct, 5 incorrect)
	incorrect	5	0	
Don't know		2	0	2
TOTALS		10	9	

Explanation: Focused on meaning. Four teacher candidates gave answers that focused on what division by zero means. Two approaches were used: (a) showing that division by zero was undefined and (b) showing that the quotient "explodes" as the divisor decreases. Tim, a mathematics major, chose the first approach, showing that division by zero was undefined he said that he would write 7 ÷ 0 "in mathematical form" on the board--that is, with the division bracket: $0\overline{)7}$. Then he would explain that

> You cannot divide 7 by 0 because there is nothing multiplied by 0 to get 7. In other words, everything multiplied by a 0 is 0, so if we had 0 divided by 7 there is nothing multiplied, there is no number up here you could put to get 0. There's no number you can put up here to get 7. And I would show them that. Whereas 6 divided by 2 there's a number you can put up there. And whenever you come across that case, you can't find a number to put up there, it doesn't exist, you can't do it.

Allen, an elementary major, explained division by zero using the second approach, showing how the quotient explodes:

> Dividing 7 by 3 and then divide 7 by 2 and then divide 7 by 1 and . . . when they get up to 7 divided by 1 is *7* and you were to go one step farther, you'd have numbers what were keep getting larger and larger. . . . One step farther you would have to . . . say divide it by 0, because dividing by decimals or fractions of that type. . . . Then you'd start, I guess it would be better to start getting closer to zero using the decimals and see that dividing 7 by fractions makes numbers . . . keep getting larger and larger. . . . If you keep making . . . the divisor closer and closer to 0, the number's just gonna keep getting larger and larger and *larger*. . . . Then I'd start asking them what the largest number they can think of is so then, that there is no largest number that . . . there is really no . . . such statement as 7 divided by 0.

Tim and Allen both focused specifically on the case of dividing by zero. What their answers had in common was the aim of showing why the particular case of division by zero is impossible. Their explanations were mathematical ones, not the "explanations" used by many of the teacher candidates, which consisted largely of restating rules.

"You can't divide by zero." Seven teacher candidates explained division by zero in terms of a rule such as "you can't divide by zero." Unlike those who focused on meaning, these prospective teachers did not try to show why this was so. Instead they emphasized the importance of remembering the rule. Terrell, a mathematics major, said emphatically,

> I'd just say. . . . "It's undefined," and I'd tell them that this is a rule that you should never forget that anytime you divide by 0 you can't. You just can't. It's undefined, so . . . you just can't.

He added, "Anytime you get a number divided by zero, then you did something wrong before." Andy, another mathematics major, said, "You can't divide by zero . . . it's just something to remember." Cindy, also a math major, said she would tell students that "this is something that you won't ever be able to do in mathematics"--even in calculus.

"Anything divided by 0 is 0." Five other teacher candidates responded in term of a rule. Like the prospective teachers quoted above, their notions of "explanation" in mathematics seemed to mean restating rules. What made their responses different, however, was that the rule they invoked was not true. Linda, an elementary major, was perhaps the most emphatic:

> I'd just say, "Anything divided by 0 is 0. That's just a rule, you just *know* it." Or I'd say, "Well, if you don't have anything, you can't get anything out. You know, it's empty, it's

> nothing. . . . "Anything multiplied by 0 is 0. I'd just say, "That's something that you have to learn, you have to know." I think that's how I was told. You just know it. . . . If they were older and they asked me "Why?" I'd just have to start mumbling about something . . . I don't know what. I'd just tell them "Because!" (laughs). . . . That's just one of those rules . . . something like in English . . . sometimes the C sounds like K or . . . you just learn, I before E except after C.

Interestingly, although Linda mentioned multiplication by zero, she doesn't connect that understanding (n x 0 = 0) with the problem of dividing by zero. Like those who stated "you can't divide by zero," these prospective teachers all emphasized the absoluteness of the rule and the value of getting pupils to remember it. Explaining and knowing mathematics were reduced to stating rules (Ball, in press-a). However, these teacher candidates did not realize that what they were saying was not true.

"I don't remember." Two prospective elementary teachers said they could not remember the answer to 7 ÷ 0. Mei Ling said simply, "7 divided by 0? Isn't that--isn't there a term for the answer to that? I can't remember." Rachel, who had taken a little more math, more recently and more successfully than most of the other elementary majors, was simply stumped by this question. "Seven divided by 0," she mused. "I'm having trouble . . . is that 0 or is that 7? I'm trying to think myself."

Teacher candidates' knowledge about division by zero. Division by zero comes up frequently in college mathematics; math majors have had more and more recent experience with dividing by zero than have nonmath majors. As such, it was not surprising that the secondary candidates were better prepared than their elementary counterparts to deal with this question, both in terms of providing mathematical explanations and in terms of knowing the correct rule. Still, most of the teacher candidates, whether right or wrong, whether focused on meaning or on rules, did not seem to refer to the more general concept of division to provide their explanations. Instead they recognized division by zero as a particular case for which there was a rule. Their explanations were simply statements of what they though to be the rule for this specific case. Furthermore, half the elementary candidates had the rule wrong. Because they did not think about the meaning of division by zero, they did not monitor the reasonableness of their answers.

Knowledge of Division in Algebraic Equations

A third interview question provided yet one more angle on the teacher candidates' understanding of division:

Suppose that one of your students asks you for help with the following exercise:

if $\frac{x}{0.2} = 5$*, then* $x=$

How would you respond?
Why is that what you'd do?[8]

In algebra classes, students are taught procedures for "isolating x"--that is, for manipulating equations so that the unknown quantity is on one side of the statement and a number is on the other. This enables one to solve the equation, or figure out what number(s) x could be. For example, the ubiquitous procedural script for solving the equation discussed above is:

You want to isolate x, so you want to get rid of the point 2 in the denominator.

Multiply both sides by point 2.

$$(.2)\ \frac{x}{0.2} = 5\ (.2)$$

The point 2's cancel on the left side; 5 times point 2 is 1. So x is 1.

Learning procedures such as these often seems to eclipse any focus on the meaning of the equations or the numbers. Furthermore, referring to .2 as "point two" does not emphasize the meaning of the number as *two-tenths*.

[8] This question was presented to elementary as well as secondary teacher candidates. Its function in the interview was to extend the analysis of their understanding of division in different contexts. In other words, was division with fractions one case, division by zero another, and division in algebra something yet entirely different again? Lest critics argue that this content is too advanced for the elementary teacher candidates, I contend that it is not unreasonable to expect that teachers whose Michigan teaching certificate will extend through eighth grade in all subjects should understand division in simple algebraic equations.

Scripts similar to this one were what the teacher candidates produced in response to this question. Overwhelmingly the teacher candidates "explained" it by restating the steps of procedures to solve such equations. Only one prospective teacher talked about it in terms of what it meant, and a few teacher candidates did not know how to do it at all. (See Table 3 for teacher candidates' responses.)

Focus on meaning. Only one teacher candidate--an elementary major--tried to talk about the meaning of the equation. Sandi said that she would want the pupil "to understand what he's doing first." She said she would help the pupil understand "the idea that the .2 has to go into x." While her explanation was vague, she was trying to make sense of the problem by reasoning about division.

Focus on procedures. Fourteen of the prospective teachers, including all of the mathematics majors, focused on the mechanics of manipulating algebraic equations. Terrell, a secondary candidate, said

> I'd explain that somehow you have to get this x by itself without that .02, I mean 0.2 . . . and then I'd ask her . . . I'd tell her somehow she's going to have to get rid of that .02.

Then he laughed self-consciously--"the complex math terms that teachers use, like `get rid of.'" The other teacher candidates gave similar answers. They all talked about getting "rid of" the .2, isolating x, and multiplying both sides by .2. They seemed to see the question as quite straightforward and simple, unlike some of the other questions I had asked, probably because solving simple equations was something they had done themselves many times and they could, for the most part, remember how to do it.

"I have no idea!" Four elementary teacher candidates did not know how to solve the equation themselves. One was overwhelmed at the prospect of having to help a student solve an equation such as this one. "Oh, my *God*!" she exclaimed when I presented her with the question. She said she had no idea, although she knew "there's steps that you go through to do it." Another said she hadn't "done these" in so long that she just couldn't remember.

All four of the teacher candidates who could not solve the equation attributed it to not having done algebra problems in a long time and not being able to remember the procedures for solving equations such as this one. The only difference between these teacher candidates and the 14 who focused on procedures was that these 4 could not remember the procedures. However, like the 13, they did not focus on the *meaning* of the mathematical statement.

Table 3

Division in Algebraic Equations: $\frac{x}{0.2} = 5$

Teacher Candidates (N=19)

Category of Response		Elementary	Secondary	TOTALS
	Meaning	1	0	1
	Procedure	5	9	14
	Don't Know	4	0	4
	TOTALS	10	9	

Table 4

Summary: Qualitative Dimensions of the Prospective Teachers' Knowledge of Division

	Truth Value	Legitimacy	Connectedness
Division of Fractions	12 [6 / 6]	5 [0 / 5]	1 [0 / 1]
Division of zero	12 [3 / 9]	5 [1 / 4]	1 [0 / 1]
Division with algebraic equations	15 [6 / 9]	1 [1 / 0]	0 [ ]

Note: Numbers are out of a possible total of 19; each cell is tabulated independently. Numbers in inserts represent distribution of responses by elementary and secondary teacher candidates: E/S.

Summary: The Prospective Teachers' Knowledge of Division

Table 4 summarizes the teacher candidates' understanding along the three qualitative dimensions of substantive knowledge: truth value, legitimacy, and connectedness. Many more students were able to gave answers that were correct (e.g., "Division by zero is undefined") than were able to explain those answers legitimately (e.g., to explain what it means for something to be "undefined"--beyond "you can't do it"). In only a couple of cases was there clear evidence that students' knowledge of division was connected across the three contexts. And a significant number of students did not even produce correct answers. Although the three interview questions all dealt with division, the teacher candidates did not focus from case to case on the concept of division. Instead, most of them responded to each question in terms of the specific bit of mathematical knowledge entailed--division of fractions, division by zero, solving algebraic equations involving division. For all three questions, the prospective teachers, both the mathematics majors and the elementary candidates, tended to search for the particular rules--"you can't divide by 0" or "get rid of the denominator"--rather than focusing on the underlying meanings of the problems presented. There are two possible influences on these findings.

Confounding of remembering and understanding. Why were the teacher candidates' responses so overwhelmingly fact and rule-oriented? Was the preponderance of procedural answers influenced by the nature of the questions themselves? Two of the questions--division by zero and division in algebraic equations--were formulated in such a way that teacher candidates could simply retrieve the correct piece of information (e.g., "division by zero is undefined"), as it was taught, from mathematics memory storage. These two questions examined "conventionally packaged" pieces of knowledge--knowledge that the teacher candidates had been taught in school. If they could remember the necessary piece, they could answer each question by stating the rule. In fact, many of them equated remembering with knowing.

One might argue that nothing in either question compelled them to talk about meaning, nor encouraged them to access legitimate explanations; however, both questions did ask the teacher candidates how they would respond to a pupil who raised that question. The dominance of procedural answers would suggest that the prospective teachers favored giving pupils rules to accept and remember, rather than conceptual explanations. However, there was substantial evidence in their responses across the interviews that the teacher candidates wanted to give the pupils more meaningful answers but could not do so, that their subject matter knowledge, lacking mathematical legitimacy, was insufficient to act on that commitment. One of the math majors realized this and commented (about division by zero), "I just *know* that . . . I don't really know why . . . it's almost become a fact . . . something that it's just there."

When answering the questions, many of them agonized over not having a "concrete example" or not knowing why something was true. One of the math majors, for example, in answering the division by zero question, said she "would hate to say it is one of those things that you have to accept in math" but that she might have to in this case if she couldn't think of a concrete example. Another laughed wryly at himself for using the phrase "get rid of the denominator," but did not have accessible any alternative ways of understanding. The answers the teacher candidates gave--rules--were what they understood, what they remembered from what their teachers said.

Moreover, some of the teacher candidates could not remember the rules at all. Once forgotten, rules are not easily retrievable without the concepts to support them (Hiebert and Lefevre, 1986). Mere remembering only serves one well in displaying mathematical knowledge--until one forgets, that is. The prospective teachers' knowledge seemed founded more on memorization than on conceptual understanding. The secondary teacher candidates, having had more (and more recent) opportunities to maintain their inventory of remembered knowledge, were therefore more likely to have something to say, less likely to draw a complete blank.

Fragmented understanding. The prospective teachers' focus on the surface differences among the three cases of division suggests that their understanding comprised remembering the rules for specific cases, not a web of interconnected ideas. Evidence for this is especially clear in the teacher candidates' efforts to generate representations for 1 3/4 ÷ ½. This task, unlike the other two division questions, did require them to do more than reproduce what they had been taught. Division with fractions is rarely taught conceptually in school; most of the prospective teachers probably learned to divide with fractions without necessarily thinking about what the problems meant. Indeed, most of them could carry out the procedure to produce the correct answer--a task that required them to remember and use the rule "invert and multiply."

Yet, when they tried to generate a representation for the statement, most of them either represented 1 3/4 ÷ 2 or couldn't do it at all. Only 5 out of the 19 teacher candidates talked about "how many halves are in 1 3/4." The results for this question suggest that, in almost all cases, the prospective teachers' understanding of division with fractions consisted of remembering a particular rule and was unattached to other ideas about division. The results for the other two questions (division by zero and division in algebraic equations) are consistent with this interpretation. It is not surprising that these students conceived of mathematics in this way. The standard school mathematics curriculum, to which most prospective teachers have been subjected, treats ideas as discrete bits of procedural knowledge, a point worth noting for it underscores what prospective teachers bring and what they, in many cases, must overcome in learning to teach even "simple" concepts like division.

Prospective Teachers' Substantive Knowledge of Mathematics: The Need to Confront Common Assumptions

Examining prospective teachers' substantive knowledge of mathematics raises serious questions about subject matter preparation for mathematics teaching. I return to the three assumptions with which this paper opens.

First Assumption: Traditional School Mathematics Content Is Simple

To assume that the content of first-grade mathematics is something any adult understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized. Throughout the interviews, many college students, including people who were majoring in mathematics, had difficulty working below the surface of so-called simple mathematics. Although they could perform the procedures, they seemed to lack warranted understanding of the content.

Close analyses of the mathematics entailed in division of fractions, of zero, and in algebra show that elementary content, if taken seriously, is anything but simple. Duckworth (1987) refers to the "depths and perplexities of elementary arithmetic" and in her writing, as well as in Lampert's (1985, 1986, in press), the "simple" content of the school curriculum is opened up and its mathematical complexity revealed. Teacher educators may be able to convince prospective teachers that "teaching for conceptual understanding" should be the goal. However, without revisiting the "simple" mathematical content they will teach--to revise and develop correct understandings of the underlying principles and warrants, of the connections among ideas--prospective teachers may be wholly unprepared to do more than teach "invert and multiply."

The Second Assumption: Elementary and Secondary School Math Classes Can Serve As Subject Matter Preparation for Teaching Mathematics

The findings discussed in this paper challenge the assumption that prospective teachers' school mathematics education can constitute sufficient subject matter preparation for teaching. In order to respond to the interview questions and tasks, the teacher candidates drew on what they had learned in school. When they did this, seeking particular mathematical concepts, procedures, or even terms, they typically found loose fragments--rules, tricks, and definitions--without warrants and unconnected. Most did not find meaningful understanding, nor even the "stuff" to figure out such understandings on the spot.

While troubling, these results should not be surprising. The widely criticized algorithmic knowledge fostered in many math classrooms is well documented (e.g., Davis and Hersh, 1981;

Erlwanger, 1975; Goodlad, 1984; Madsen-Nason and Lanier, 1987; Wheeler, 1980). Recent results of the National Assessment of Educational Progress (Dossey, Mullis, Lindquist, and Chambers, 1988) suggest that many students do not develop deep and principled understandings of mathematics; the achievement data on 17-year-olds is particularly alarming. The findings here simply reinforce what these data ought to suggest: that relying on what prospective teachers have learned in their precollege mathematics classes is unlikely to be adequate for teaching mathematics for understanding.

The Third Assumption: Majoring in Mathematics Ensures Subject Matter Knowledge

I have been finding narrower differences in substantive understanding of mathematics between elementary and secondary teacher candidates than one might expect (or hope). The latter, because they are math majors, have taken more mathematics and do know more stuff--that is, they get more answers right (although they also are wrong a significant amount of the time). Still, their additional studies do not seem to afford them substantial advantage in explaining and connecting underlying concepts, principles, and meanings.

Some analysts propose that this is due largely to the poor academic caliber of teacher education students (see Lanier, 1986, for a refutation of this common assertion). However, interviews conducted by researchers at the National Center for Research on Teacher Education with mathematics majors who are not planning to teach do not support this suggestion. These math majors, too, struggle with making sense of division with fractions, connecting mathematics to the real world, and coming up with explanations that go beyond the restatement of rules. Furthermore, most of the secondary teacher candidates in this study were good students, with impressive college entrance exam scores and high grade point averages in their college math courses.

A more plausible explanation for the problems experienced by the math majors is that even successful participation in traditional math classes does not necessarily develop the kinds of understanding needed to teach if, as is often the case, success in these classes derives from memorizing formulas and performing procedures. Moreover, studying calculus does not usually afford students the opportunity to revisit or extend their understandings of arithmetic, algebra, or geometry, the subjects they will teach. Requiring teachers to major in mathematics, or even increasing the mathematics course requirements for prospective teachers, both currently advocated, will not necessarily ensure increases in their substantive understanding.

Conclusion

Although subject matter knowledge is widely acknowledged as an essential component of teacher knowledge, the subject matter preparation of teachers is rarely the central focus of any phase of teacher education. Instead, everyone is willing to assume that it will happen somewhere else: prior to college, in liberal arts classes, from teaching. What we are learning about the understandings of mathematics that prospective teachers bring with them to teacher education suggests the danger of assuming that subject matter preparation will indeed happen "somewhere else" and points to the need to make it a central focus. Doing that requires not only changes in emphasis since much mathematics teaching does not produce the kind of understandings of mathematics that teachers need. Attending seriously to the subject matter preparation of elementary and secondary math teachers implies the need to know much more than we currently do about how teachers can be helped to transform and increase their understandings of mathematics, working with what they bring and helping them move toward the kinds of mathematical understanding needed in order to teach mathematics well.

References

Ball, D. L. (1988). *Knowledge and reasoning in mathematical pedagogy: Examining what prospective teachers bring to teacher education*. Unpublished doctoral dissertation. Michigan State University, East Lansing.

Ball, D. L. (in press-a). Prospective teachers' knowledge of mathematics. *Elementary School Journal*.

Ball, D. L. (in press-b). Research on teaching mathematics: Making subject matter knowledge part of the equation. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 2. Teachers' subject matter knowledge and classroom instruction*. Greenwich, CT: JAI Press.

Bogdan, R., and Biklen, S. K. (1982). *Qualitative research for education: An introduction to theory and methods*. Boston: Allyn and Bacon.

Buchmann, M. (1984). The priority of knowledge and understanding in teaching. In J. Raths and L. Katz (Eds.), *Advances in teacher education* (Vol. 1, pp. 29-48). Norwood, NJ: Ablex.

Carnegie Task Force on Teaching as a Profession. (1986). *A nation prepared: Teachers for the 21st century*. Washington, DC: Carnegie Forum on Education and the Economy.

Davis, P., and Hersh, R. (1981). *The mathematical experience*. New York: Houghton Mifflin.

Davis, R. (1983). Diagnosis and evaluation in mathematics education. In D. Smith (Ed.), *Essential knowledge for beginning educators* (pp. 101-111). Washington, DC: American Association of Colleges for Teacher Education

Davis, R. (1986). Conceptual and procedural knowledge in mathematics: summary analysis. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 265-300). Hillsdale, NJ: Erlbaum.

Dossey, J., Mullis, I., Lindquist, M., and Chambers, D. (1988). *The mathematics report card: Are we measuring up? Trends and achievement based on the 1986 National Assessment*. Princeton, NJ: Educational Testing Service.

Duckworth, E. (1987). Some depths and perplexities of elementary arithmetic. *Journal of Mathematical Behavior, 6*, 43-94.

Erlwanger, S. (1975). Benny's conceptions of rules and answers in IPI mathematics. *Journal of Children's Mathematical Behavior, 1*, 157-283.

Goodlad, J. I. (1984). *A place called school*. New York: McGraw-Hill.

Graeber, A., Tirosh, D., and Glover, R. (1986, September). *Preservice teachers' beliefs and* *performance on measurement and partitive division problems*. Paper presented at the 8th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, East Lansing, MI.

Greeno, J. (1978). Understanding and procedural knowledge in mathematics education. *Educational Psychologist*, *12*, 262-283.

Hatano, G. (1982). Cognitive consequences of practice in culture-specific procedural skills. *The Quarterly Newsletter of the Laboratory of Comparative Human Cognition*, *4*(1), 15-18.

Hiebert, J., and Lefevre, P. (1986). Conceptual and procedural knowledge. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1-27). Hillsdale, NJ: Erlbaum.

Holmes Group. (1986). *Tomorrow's teachers: A report of the Holmes Group*. East Lansing: Michigan State University, College of Education.

Kline, M. (1980). *Mathematics: The loss of certainty*. Oxford: Oxford University Press.

Lakatos, I. (1976). *Proofs and refutations*. Cambridge, England: Cambridge University Press.

Lampert, M. (1985). Mathematics learning in context: The Voyage of the Mimi. *The Journal of Mathematical Behavior, 4,* 157-167.

Lampert, M. (1986). Knowing, doing, and teaching multiplication. *Cognition and Instruction*, *3*, 305-342.

Lampert, M. (1988, July). *From invention to convention: The teacher's role in classroom communication about mathematics*. Paper presented at International Conference for Mathematics Education, Budapest, Hungary.

Lampert, M. (in press). Choosing and using mathematical tools in classroom discourse. In J. Brophy (Ed.), *Advances in research on teaching: Vol. 1. Teaching for meaningful understanding and self-regulated learning*. Greenwich, CT: JAI Press.

Lanier, J. (1986). Research on teacher education. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 527-569). New York: Macmillan.

Madsen-Nason, A., and Lanier, P. (1986). *Pamela Kaye's general math class: From a computational to a conceptual orientation* (Research Series No. 172). East Lansing: Michigan State University, Institute for Research on Teaching.

Nickerson, R. (1985). Understanding understanding. *American Journal of Education*, *93*, 201-239.

Noddings, N. (1985). Formal modes of knowing. In E. Eisner (Ed.), *Learning and teaching and the ways of knowing,* (84th yearbook of the National Society for the Society of Education, pp. 116-132). Chicago: University of Chicago Press.

Orr, E. W. (1987). *Twice as less: Black English and the performance of black students in mathematics and science*. New York: Norton.

Resnick, L., and Ford, W. (1981). *The psychology of mathematics for instruction*. Hillsdale, NJ: Erlbaum.

Romberg, T. (1983). A common curriculum for mathematics. In G. Fenstermacher and J. Goodlad (Eds.). *Individual differences and the common curriculum* (82nd yearbook of the National Society for the Study of Education). Chicago: University of Chicago Press.

Scheffler, I. (1965). *Conditions of knowledge: An introduction to epistemology and education*. Chicago: University of Chicago Press.

Schon, D. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.

Schwab, J. J. (1978). Education and the structure of the disciplines. In I. Westbury and N. Wilkof (Eds.), *Science, curriculum, and liberal education: Selected essays* (pp. 229-272). Chicago: University of Chicago Press. (Original work published 1961)

Schwartz, J. (1987, October). *Use of computers in the mathematics curriculum*. Paper presented at the College of Education, Michigan State University, East Lansing.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, *15*(2), 4-14.

Skemp, R. (1978). Relational understanding and instrumental understanding. *Arithmetic Teacher*, *26*, 9-15.

Wheeler, D. (1980). An askance look at remediation in mathematics. *Outlook, 38*, 41-50.

Wilson, S. M. (1988). *Understanding historical understanding: Subject matter knowledge and the teaching of U. S. history*. Unpublished doctoral dissertation, Stanford University.

Wilson, S. M., and Ball, D. L. (in preparation). *Historical and mathematical ways of knowing: A contrast in subject matter knowledge*. National Center for Research on Teacher Education, Michigan State University, East Lansing.